Integration of Rational Functions by Partial Fractions

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Calculus II: August 2016 - December 2016

Partial Fractions

A fraction is either a proper fraction or an improper fraction. In a **proper fraction**, the degree of the numerator is strictly less the degree of the denominator. In an **improper fraction**, the degree of the numerator is greater than or equal to the degree of the denominator.

$$\text{Proper fraction}: \frac{x^2+1}{x^3+2x+4}$$
$$\text{Improper fraction}: \frac{x^3+1}{x^2+2x+1}$$
$$\text{Improper fraction}: \frac{x^2+1}{x^2+2x+1}$$

The improper fraction $\frac{9}{2}$ can be represented as 2×4 remainder 1. The same concept can be applied to polynomials.

$$\begin{array}{r|l}
 & x^3 + x^2 + x + 1 \\ \hline
x-1 & x^4 \\
 & -x^4 + x^3 \\ \hline
 & x^3 \\
 & -x^3 + x^2 \\ \hline
 & x^2 \\
 & -x^2 + x \\ \hline
 & x \\
 & -x + 1 \\ \hline
 & 1
\end{array}$$

Therefore:

$$\int \frac{x^4}{x-1}\,\mathrm{d}x$$

can be represented as:

$$\int \left(x^3+x^2+x+1+\frac{1}{x-1}\right)\mathrm{d}x$$

$$\int \left(x^3+x^2+x+1\right)\mathrm{d}x+\int \frac{1}{x-1}\,\mathrm{d}x$$

$$\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x+\ln|x-1|+C$$

For proper fractions, we must do something else. As a general rule, if we have a polynomial of the form:

$$\frac{Ax+B}{(Cx+D)(Ex+F)}$$

If we wanted to integrate this, we would have to break it up into the form:

$$\frac{Ax+B}{(Cx+D)(Ex+F)}=\frac{U}{Cx+D}+\frac{V}{Ex+F}$$

where U and V are unknown. The unknown polynomial in the numerator follows the degree of the denominator:

$$\frac{Ax+C}{(Cx^2+D)(Ex+F)}=\frac{Ux+V}{Cx^2+D}+\frac{W}{Ex+F}$$

$$\frac{Ax+B}{(Cx^3+D)(Ex+F)}=\frac{Tx^2+Ux+V}{Cx^3+D}+\frac{W}{Ex+F}$$

$$\frac{Ax+B}{(Cx^4+D)(Ex+F)}=\frac{Sx^3+Tx^2+Ux+V}{Cx^4+D}+\frac{W}{Ex+F}$$

And so on. Note that you should avoid using C as a variable for a coefficient if the problem involves integration so that you do not confuse it with the constant of integration.

For this example, both the terms in the denominator are of degree 1, so the polynomial in the numerator will be of degree 0:

$$\frac{4+x}{(1+2x)(3-x)}$$

$$\frac{4+x}{(1+2x)(3-x)} = \frac{A}{1+2x} + \frac{B}{3-x}$$

If we take A and B as our unknowns, we can solve for them in terms of x by cross multiplying.

$$(1+2x)(3-x)\frac{4+x}{(1+2x)(3-x)} = (\frac{A}{1+2x} + \frac{B}{3-x})(1+2x)(3-x)$$

$$4+x = A(3-x) + B(1+2x)$$

Method I

$$4+x = A(3-x) + B(1+2x)$$

$$4+x = 3A - Ax + B + 2Bx$$

$$4+x = (3A+B) + (2B-A)x$$

$$4+(1)x = (3A+B) + (2B-A)x$$

$$4 = 3A+B \quad 1 = 2B-A$$

$$6A+2B = 8$$

$$-A+2B = 1$$

$$7A = 7 \quad A = 1$$

$$B = 1$$

Method II

$$4+x = A(3-x) + B(1+2x)$$

$$Let: x = 3$$

$$7 = 4+3 = A(3-3) + B(1+2(3))$$

$$B = 1$$

$$Let: x = \frac{-1}{2}$$

$$4 - \frac{1}{2} = A(3 - \frac{-1}{2}) + B(1 + 2\frac{-1}{2})$$

$$\frac{7}{2} = \frac{7}{2}A$$

$$A = 1$$

Therefore:

$$\frac{4+x}{(1+2x)(3-x)} = \frac{1}{1+2x} + \frac{1}{3-x}$$

$$\int \frac{4+x}{(1+2x)(3-x)}\ \mathrm{d}x = \int \frac{1}{1+2x}\ \mathrm{d}x + \int \frac{1}{3-x}\ \mathrm{d}x$$

$$2\ln|1+2x| + \ln|3-x| + C$$

$$= \ln\left(|1+2x|^2|3-x|\right) + C$$

Here is a different example with a proper fraction that has a denominator with a higher degree.

$$\frac{1}{x^2+x^4}$$

Method 1

$$\frac{1}{x^2+x^4} = \frac{1}{(x^2)(1+x^2)}$$

$$\frac{1}{(x^2)(1+x^2)} = \frac{Ax+B}{x^2} + \frac{Cx+D}{1+x^2}$$

$$1 = (Ax+B)(1+x^2) + (Cx+D)x^2$$

$$1 = Ax + B + Ax^3 + Bx^2 + Cx^3 + Dx2$$

$$1 = (A+C)x^3 + (B+D)x^2 + Ax + B$$

$$A+C=0 \quad B+D=0 \quad A=0 \quad B=1$$

$$D=-1 \quad C=0$$

$$\frac{1}{(x^2)(1+x^2)} = \frac{1}{x^2} - \frac{1}{1+x^2}$$

Method 2

$$\frac{1}{x^2+x^4} = \frac{1}{(x^2)(1+x^2)}$$

$$\frac{1}{(x^2)(1+x^2)} = \frac{1+x^2-x^2}{x^2(1+x^2)}$$

$$\frac{1+x^2}{x^2(1+x^2)} + \frac{-x^2}{x^2(1+x^2)}$$

$$= \frac{1}{x^2} - \frac{1}{1+x^2}$$

Practice Problem 2

$$\frac{x-6}{x^2+x-6}$$

$$\frac{x-6}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

$$x-6 = A(x-2) + B(x+3)$$

$$Let : x = -3$$

$$-3-6 = A(-3-2) + B(-3+3)$$

$$A = \frac{9}{5}$$

$$Let : x = 2$$

$$2-6 = A(2-2) + B(2+3)$$

$$B = \frac{-4}{5}$$

$$\frac{x-6}{(x+3)(x-2)} = \frac{9}{5(x+3)} - \frac{4}{5(x-2)}$$

Practice Problem 3a

$$\frac{x^4+1}{x^5+4x^3}$$

$$\frac{x^4+1}{x^5+4x^3} = \frac{x^4+1}{x^3(x^2+4)}$$

$$\frac{x^4+1}{x^3(x^2+4)} = \frac{Ax^2+Bx+C}{x^3} + \frac{Dx+E}{x^2+4}$$

$$\frac{x^4+1}{x^3(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+4}$$

$$x^3(x^2+4)\frac{x^4+1}{x^3(x^2+4)} = \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+4}\right)(x^3(x^2+4))$$

$$x^4+1 = Ax^4+4Ax^2+Bx^3+4Bx+Cx^2+4C+Dx^4+Ex^3$$

$$x^4+1 = (A+D)x^4+(B+E)x^3+(4A+C)x^2+4Bx+4C$$

$$A+D=1 \quad B+E=0 \quad 4A+C=0 \quad 4B=0 \quad 4C=1$$

$$A=\frac{-1}{16} \quad B=0 \quad C=\frac{1}{4} \quad D=\frac{17}{16} \quad E=0$$

$$\frac{x^4+1}{x^3(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+4}$$

$$= \frac{-1}{16x} + \frac{1}{4x^3} + \frac{17x}{16(x^2+4)}$$

Practice Problem 3b

$$\frac{1}{(x^2-9)^2}$$

$$\frac{1}{(x^2-9)^2} = \frac{1}{(x-3)^2(x+3)^2}$$

$$\frac{1}{(x-3)^2(x+3)^2} = \frac{Ax+B}{(x-3)^2} + \frac{Cx+D}{(x+3)^2}$$

$$\frac{1}{(x-3)^2(x+3)^2} = \frac{Ax}{(x-3)^2} + \frac{B}{(x-3)^2} + \frac{Cx}{(x+3)^2} + \frac{D}{(x+3)^2}$$

$$((x-3)^2(x+3)^2)\frac{1}{(x-3)^2(x+3)^2} = \left(\frac{Ax}{(x-3)^2} + \frac{B}{(x-3)^2} + \frac{Cx}{(x+3)^2} + \frac{D}{(x+3)^2}\right)((x-3)^2(x+3)^2)$$

$$1 = Ax(x+3)^2 + B(x+3)^2 + Cx(x-3)^2 + D(x-3)^2$$

$$1 = Ax(x^2+6x+9) + B(x^2+6x+9) + Cx(x^2-6x+9) + D(x^2-6x+9)$$

$$1 = Ax^3 + 6Ax^2 + 9Ax + Bx^2 + 6Bx + 9B + Cx^3 - 6Cx^2 + 9Cx + Dx^2 - 6Dx + 9D$$

$$1 = (A+C)x^3 + (6A+B-6C+D)x^2 + (9A+6B+9C-6D)x + (9B+9D)$$

$$A+C=0 \quad 6A+B-6C+D=0 \quad 9A+6B+9C-6D=0 \quad 9B+9D=1$$

$$54A+9B-54C+9D=0 \quad 9B+9D=1$$

$$54A-54C=-1 \quad A+C=0$$

$$108A=-1$$

$$A=\frac{-1}{108} \quad C=\frac{1}{108}$$

$$9A+6B+9C-6D=0=6B-6D$$

$$B-D=0 \quad B+D=\frac{1}{9}$$

$$B=\frac{1}{18} \quad D=\frac{1}{18}$$

$$\frac{1}{(x-3)^2(x+3)^2} = \frac{-x}{108(x-3)^2} + \frac{1}{18(x-3)^2} + \frac{x}{108(x+3)^2} + \frac{1}{18(x+3)^2}$$

Practice Problem 12

$$\int \frac{x-4}{x^2-5x+6}\,\mathrm{d}x$$

$$\frac{x-4}{x^2-5x+6} = \frac{x-4}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

$$\frac{x-4}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

$$x-4 = A(x-2) + B(x-3)$$

$$Let: x=3$$

$$3-4 = A(3-2) + B(3-3)$$

$$A=-1$$

$$Let: x=2$$

$$2-4 = A(2-2) + B(2-3)$$

$$B = 2$$

$$\int \frac{x-4}{x^2-5x+6}\ \mathrm{d}x = \int \frac{A}{x-3} + \frac{B}{x-2}\ \mathrm{d}x = \int \frac{A}{x-3}\ \mathrm{d}x + \int \frac{B}{x-2}\ \mathrm{d}x$$

$$A\int \frac{1}{x-3}\ \mathrm{d}x + B\int \frac{1}{x-2}\ \mathrm{d}x$$

$$A\ln|x-3| + B\ln|x-2| + C$$

$$-\ln|x-3| + 2\ln|x-2| + C$$

$$= \ln\left(\frac{|x-2|^2}{|x-3|}\right) + C$$

Practice Problem 15

$$\int \frac{x^3-4x+1}{x^2-3x+2}\ \mathrm{d}x$$

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\ \ x+3 \\ \hline x^2-3x+2 & \ x^3 \quad\quad\quad -4x+1 \\ & -x^3+3x^2-2x \\ \hline & \quad\quad 3x^2-6x+1 \\ & \quad -3x^2+9x-6 \\ \hline & \quad\quad\quad\quad 3x-5 \end{array}$$

$$\int \frac{x^3-4x+1}{x^2-3x+2}\ \mathrm{d}x = \int x+3+\frac{3x-5}{x^2-3x+2}\ \mathrm{d}x$$

$$\frac{x^2}{2} + 3x + C + \int \frac{3x-5}{x^2-3x+2}\ \mathrm{d}x$$

$$\frac{3x-5}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$

$$3x-5 = A(x-1) + B(x-2)$$

$$Let: x = 1 \to 3(1)-5 = A(1-1) + B(1-2) \to B = 2$$

$$Let: x = 2 \to 3(2)-5 = A(2-1) + B(2-2) \to A = 1$$

$$\frac{x^2}{2} + 3x + C + \int \frac{3x-5}{(x-2)(x-1)}\ \mathrm{d}x$$

$$\frac{x^2}{2} + 3x + C + \int \frac{A}{x-2} + \frac{B}{x-1}\ \mathrm{d}x$$

$$\frac{x^2}{2} + 3x + A\ln|x-2| + B\ln|x-1| + C$$
$$= \frac{x^2}{2} + 3x + 1\ln|x-2| + 2\ln|x-1| + C$$
$$= \frac{x^2}{2} + 3x + \ln\left(|x-2||x-1|^2\right) + C$$

Practice Problem 24

$$\int \frac{x^2-x+6}{x^3+3x}\,\mathrm{d}x$$
$$\int \frac{x^2-x+6}{x^3+3x}\,\mathrm{d}x = \int \frac{x^2-x+6}{x(x^2+3)}\,\mathrm{d}x$$
$$\frac{x^2-x+6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+D}{x^2+3}$$
$$x^2-x+6 = A(x^2+3) + (Bx+D)x$$
$$x^2-x+6 = Ax^2 + 3A + Bx^2 + Dx$$
$$x^2-x+6 = (A+B)x^2 + Dx + 3A$$
$$A+B=1 \quad D=-1 \quad 3A=6$$
$$A=2 \quad B=-1 \quad D=-1$$
$$\int \frac{x^2-x+6}{x(x^2+3)}\,\mathrm{d}x = \int \frac{A}{x} + \frac{Bx+D}{x^2+3}\,\mathrm{d}x$$
$$\int \frac{2}{x} + \frac{-x-1}{x^2+3}\,\mathrm{d}x$$
$$2\int \frac{1}{x}\,\mathrm{d}x - \int \frac{x+1}{x^2+3}\,\mathrm{d}x$$
$$2\ln|x| + C - \int \frac{x}{x^2+3}\,\mathrm{d}x - \int \frac{1}{x^2+3}\,\mathrm{d}x$$
$$2\ln|x| - \frac{1}{2}\ln|x^2+3| + C - \int \frac{1}{x^2+3}\,\mathrm{d}x$$

We can solve this integral by rewriting the derivative of $\tan^{-1}$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(\frac{x}{a}) = \frac{\frac{1}{a}}{1+(\frac{x}{a})^2} \times \frac{a^2}{a^2} = \frac{a}{x^2+a^2}$$

$$\int \frac{a}{x^2 + a^2} \, \mathrm{d}x = \tan^{-1}(\frac{x}{a}) + C$$

Substitute this general form into the integral:

$$\int \frac{1}{x^2 + 3} \, \mathrm{d}x = \frac{1}{\sqrt{3}} \int \frac{\sqrt{3}}{x^2 + 3} \, \mathrm{d}x = \frac{1}{\sqrt{3}} \tan^{-1}(\frac{x}{\sqrt{3}}) + C$$

Therefore:

$$\begin{aligned} &2 \ln |x| - \frac{1}{2} \ln |x^2 + 3| + C - \int \frac{1}{x^2 + 3} \, \mathrm{d}x \\ &= 2 \ln |x| - \frac{1}{2} \ln |x^2 + 3| - \frac{1}{\sqrt{3}} \tan^{-1}(\frac{x}{\sqrt{3}}) + C \end{aligned}$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech