Trigonometric Substitution

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Trigonometric Substitution

$$\int \sin^4(x)\cos^3(x)\ \mathrm{d}x$$

When you have a product of sin and cos of different powers, you have three different possibilities:

- They are both even powers.
- They are both odd powers.
- One exponent is odd and the other is even.

We can rewrite this problem as:

$$\int \sin^4(x)\cos^2(x)\cos(x)\ \mathrm{d}x$$

We want even powers of

$$\int \sin^4(x)(1-\sin^2(x))^2(x)\cos(x)\ \mathrm{d}x$$

Now we can use substitution:

$$Let : \sin(x) = t$$

$$\cos(x) = \frac{\mathrm{d}t}{\mathrm{d}x} \quad \cos(x)\ \mathrm{d}x = \ \mathrm{d}t$$

$$\int t^4(1-t2)\ \mathrm{d}t$$

$$\int t^4 - t^6 \text{ d}t = \int t^4 \text{ d}t - \int t^6 \text{ d}t$$

$$\frac{t^5}{5} - \frac{t^7}{7} + C$$

$$\frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C$$

And in the wise words of Professor Khan: "These terms are like Hillary and Trump supporters and we cannot combine them."

Another case is where we have a difficult term inside a radical in the denominator.

$$\int \frac{\text{d}x}{x^2\sqrt{4-x^2}}$$

Note that the term in the radical has a form similar to the trigonometric identities above.

$$Let : x = 2\sin(\theta)$$

$$\text{d}x = 2\cos(\theta) \text{ d}\theta$$

$$\int \frac{2\cos(\theta) \text{ d}\theta}{(2\sin(\theta))^2\sqrt{4 - 4\sin^2(\theta)}}$$

By substituting for $2\sin(\theta)$, we can turn the radical into the form of a trigonometric identity.

$$\int \frac{2\cos(\theta)}{4\sin^2(\theta)\sqrt{4}\sqrt{1 - 1\sin^2(\theta)}} \text{ d}\theta$$

In this case, we are using the identity $\sin^2(\theta) + \cos^2(\theta) = 1$ which we can rewrite as $\cos^2(\theta) = 1 - \sin^2(\theta)$.

$$\frac{1}{4}\int \frac{\cos\theta}{\sin^2(\theta)\sqrt{cos^2(\theta)}} \text{ d}\theta$$

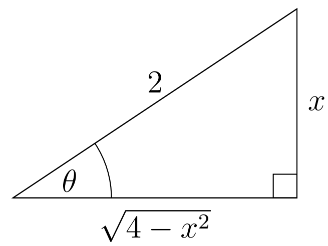
$$\frac{1}{4}\int \frac{\cos\theta}{\sin^2(\theta)cos(\theta)} \text{ d}\theta$$

$$\frac{1}{4}\int \frac{1}{\sin^2(\theta)} \text{ d}\theta$$

$$\frac{1}{4}\int \csc^2(\theta) \text{ d}\theta$$

$$\frac{1}{4}\cot(\theta) + C$$

To substitute back, we must imagine a triangle with angle θ. Given our first subsitution $x = 2\sin(\theta)$, we can rewrite it as $\sin(\theta) = \frac{x}{2} = \frac{opp}{hyp}$. If our triangle has opposite side x and hypotenuse 2, then the adjacent side must be $\sqrt{4-x^2}$.



Therefore: $\cot(\theta) = \frac{adj}{opp} = \frac{\sqrt{4-x^2}}{x}$

$$\frac{1}{4}\cot(\theta) + C = \frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$$

$$= \frac{\sqrt{4-x^2}}{4x} + C$$

Practice Problem 4

$$\int \frac{x^2}{\sqrt{9-x^2}}\ \mathrm{d}x$$

$$Let : x = 3\sin(\theta)$$

$$\mathrm{d}x = 3\cos(\theta)\ \mathrm{d}\theta$$

$$\int \frac{(3\sin(\theta))^2}{\sqrt{9-(3\sin(\theta))^2}} 3\cos(\theta)\ \mathrm{d}\theta$$

$$\int \frac{27\sin^2(\theta)\cos(\theta)}{\sqrt{9-9\sin^2(\theta)}}\ \mathrm{d}\theta$$

$$\int \frac{27\sin^2(\theta)\cos(\theta)}{\sqrt{9}\sqrt{1-1\sin^2(\theta)}}\ \mathrm{d}\theta$$

$$9\int \frac{\sin^2(\theta)\cos(\theta)}{\sqrt{cos^2(\theta)}}\ \mathrm{d}\theta$$

$$9\int \frac{\sin^2(\theta)\cos(\theta)}{cos(\theta)}\ \mathrm{d}\theta$$

$$9\int \sin^2(\theta)\ \mathrm{d}\theta$$

Using the double angle formulas:

$$9\int \left[\frac{1}{2}-\frac{1}{2}\cos(2\theta)\right]\ \mathrm{d}\theta$$

$$\frac{9}{2}\int 1-\cos(2\theta)\ \mathrm{d}\theta$$

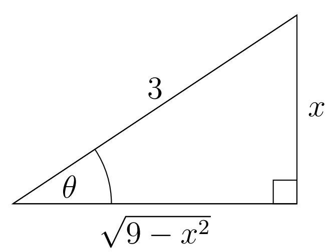
$$\frac{9}{2}\left[\theta-\frac{\sin(2\theta)}{2}\right]+C$$

Using the double angle formulas again:

$$\frac{9}{2}\left[\theta-\frac{2\sin(\theta)\cos(\theta)}{2}\right]+C$$

$$=\frac{9}{2}\left[\theta-\sin(\theta)\cos(\theta)\right]+C$$

Recall that we substituted $x=3\sin(\theta)$, which we can rewrite as $\sin(\theta)=\frac{x}{3}=\frac{opp}{hyp}$. If we imagine a triangle in which the opposite side is x and the hypotenuse is 3, then the adjacent side must be $\sqrt{9-x^2}$.



Therefore: $\cos(\theta)=\frac{adj}{hyp}=\frac{\sqrt{9-x^2}}{3}$ and $\theta=\sin^{-1}(\frac{x}{3})$

$$\frac{9}{2}\left[\theta-\sin(\theta)\cos(\theta)\right]+C=\frac{9}{2}\left[\sin^{-1}(\frac{x}{3})-\frac{x}{3}\frac{\sqrt{9-x^2}}{3}\right]+C$$

$$=\frac{9}{2}\left[\sin^{-1}(\frac{x}{3})-\frac{x\sqrt{9-x^2}}{9}\right]+C$$

Practice Problem 6

$$\int_0^3 \frac{x}{\sqrt{36-x^2}}\ \mathrm{d}x$$

$$Let: x = 6\sin(\theta)$$

$$\mathrm{d}x = 6\cos(\theta)\ \mathrm{d}\theta$$

For now, we will solve the problem as an indefinite integral.

$$\int \frac{6\sin(\theta)}{\sqrt{36-(6\sin(\theta))^2}} 6\cos(\theta)\ \mathrm{d}\theta$$

$$\int \frac{36\sin(\theta)\cos(\theta)}{\sqrt{36-36\sin^2(\theta)}}\ \mathrm{d}\theta$$

$$\int \frac{36\sin(\theta)\cos(\theta)}{\sqrt{36}\sqrt{1-1\sin^2(\theta)}}\ \mathrm{d}\theta$$

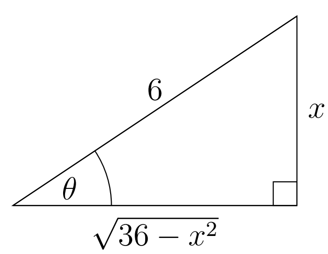
$$6\int \frac{\sin(\theta)\cos(\theta)}{\sqrt{\cos^2(\theta)}}\ \mathrm{d}\theta$$

$$6\int \frac{\sin(\theta)\cos(\theta)}{\cos(\theta)}\ \mathrm{d}\theta$$

$$6\int \sin(\theta)\ \mathrm{d}\theta$$

$$-6\cos(\theta) + C$$

Recall that we substituted $x = 6\sin(\theta)$, which we can rewrite as $\sin(\theta) = \frac{x}{6} = \frac{opp}{hyp}$. If we imagine a triangle in which the opposite side is x and the hypotenuse is 6, then the adjacent side must be $\sqrt{36-x^2}$.



Therefore: $\cos(\theta) = \frac{adj}{hyp} = \frac{\sqrt{36-x^2}}{6}$

$$-6\cos(\theta) + C = -6\frac{\sqrt{36-x^2}}{6} + C$$

$$= -\sqrt{36-x^2} + C$$

Now we can use the original limits of the intergral to solve this.

$$\Big[-\sqrt{36-x^2} \Big]_0^3$$

$$-\sqrt{36-3^2} - (-\sqrt{36-0^2})$$
$$-\sqrt{27} + 6$$
$$= 6 - 3\sqrt{3}$$

Practice Problem 8

$$\int \frac{\mathrm{d}t}{t^2\sqrt{t^2-16}}$$

Note that the radical is of the form $t^2 - 16$. We cannot subsitute $\sin(\theta)$ into this since it will not satisfy the trigonometric identity.

$$Let : x = 4\sec(\theta)$$

$$\mathrm{d}x = 4\sec(\theta)\tan(\theta)\ \mathrm{d}\theta$$

$$\int \frac{4\sec\theta\tan(\theta)}{16\sec^2(\theta)\sqrt{(4\sec(\theta))^2 - 16}}\ \mathrm{d}(\theta)$$

$$\frac{1}{4}\int \frac{\tan(\theta)}{\sec(\theta)\sqrt{16}\sqrt{\sec^2(\theta)-1}}\ \mathrm{d}(\theta)$$

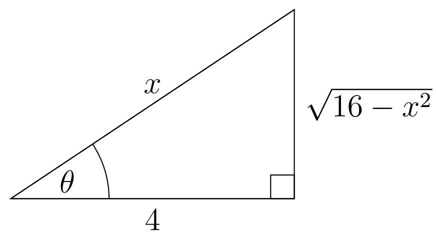
$$\frac{1}{16}\int \frac{\tan(\theta)}{\sec(\theta)\sqrt{\tan^2(\theta)}}\ \mathrm{d}(\theta)$$

$$\frac{1}{16}\int \frac{\tan(\theta)}{\sec(\theta)\tan(\theta)}\ \mathrm{d}(\theta)$$

$$\frac{1}{16}\int \cos(\theta)\ \mathrm{d}\theta$$

$$\frac{1}{16}\sin(\theta) + C$$

Recall that we substituted $x = 4\sec(\theta)$, which we can rewrite as $\sec(\theta) = \frac{x}{4} = \frac{hyp}{adj}$. If we imagine a triangle in which the hypotenuse is x and the adjacent side is 4, then the opposite side must be $\sqrt{16 - x^2}$.



Therefore: $\sin(\theta) = \frac{opp}{hyp} = \frac{\sqrt{16-x^2}}{4}$

$$\frac{1}{16}\sin(\theta) + C = \frac{1}{16}\frac{\sqrt{16-x^2}}{4} + C$$

$$\frac{\sqrt{16-x^2}}{64} + C$$

Practice Problem 15

$$\int x^2\sqrt{a^2 - x^2}\ \mathrm{d}x$$

$$Let : x = a\sin(\theta)\ \mathrm{d}\theta$$

$$\mathrm{d}x = a\cos(\theta)\ \mathrm{d}\theta$$

$$\int a^2\sin^2(\theta)\sqrt{a^2 - a^2\sin^2(\theta)}a\cos(\theta)\ \mathrm{d}\theta$$

$$a^4\int \sin^2(\theta)\cos(\theta)\sqrt{1-\sin^2(\theta)}\ \mathrm{d}\theta$$

$$a^4\int \sin^2(\theta)\cos(\theta)\sqrt{\cos^2(\theta)}\ \mathrm{d}\theta$$

$$a^4\int \sin^2(\theta)\cos^2(\theta)\ \mathrm{d}\theta$$

$$a^4\int (\sin(\theta)\cos(\theta))^2\ \mathrm{d}\theta$$

Using the double angle formulas:

$$a^4 \int (\frac{\sin(2\theta)}{2})^2 \, \mathrm{d}\theta$$

$$\frac{a^4}{4} \int \sin^2(2\theta) \, \mathrm{d}\theta$$

Using the double angle formulas again:

$$\frac{a^4}{4} \int \frac{1 - \cos(4\theta)}{2} \, \mathrm{d}\theta$$

$$\frac{a^4}{8} \int 1 - \cos(4\theta) \, \mathrm{d}\theta$$

$$\frac{a^4}{8} \left[\theta - \frac{1}{4} \sin(4\theta) \right] + C$$

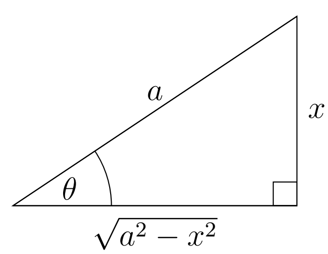
$$\frac{a^4}{8} \left[\theta - \frac{1}{4} 2 \sin(2\theta) \cos(2\theta) \right] + C$$

$$\frac{a^4}{8} \left[\theta - \frac{1}{2} (2 \sin(\theta) \cos(\theta))(1 - 2 \sin^2(\theta)) \right] + C$$

$$\frac{a^4}{8} \left[\theta - \sin(\theta) \cos(\theta) - 2 \sin^2(\theta) \sin(\theta) \cos(\theta) \right] + C$$

$$\frac{a^4}{8} \left[\theta - \sin(\theta) \cos(\theta) - 2 \sin^3(\theta) \cos(\theta) \right] + C$$

Recall that we substituted $x = a\sin(\theta)$, which we can rewrite as $\sin(\theta) = \frac{x}{a} = \frac{opp}{hyp}$. If we imagine a triangle in which the opposite side is x and the hypotenuse is a, then the adjacent side must be $\sqrt{a^2 - x^2}$.



Given this information:

$$\theta = \sin^{-1}(\frac{x}{a})$$

$$\sin(\theta) = \frac{x}{a}$$

$$\cos(\theta) = \frac{\sqrt{a^2 - x^2}}{a}$$

We can substitute this back into our solution:

$$\frac{a^4}{8}\left[\theta - \sin(\theta)\cos(\theta) - 2\sin^3(\theta)\cos(\theta)\right] + C$$

$$\frac{a^4}{8}\left[\sin^{-1}(\frac{x}{a}) - \frac{x\sqrt{a^2 - x^2}}{a^2} - \frac{2x^3\sqrt{a^2 - x^2}}{a^4}\right] + C$$

Practice Problem 21

$$\int \frac{x^2}{\sqrt{9 - 25x^2}}\ \mathrm{d}x$$

The terms inside the radical are not of the same form as the problems before. We can rewrite this problem to figure out the substitution.

$$\int \frac{x^2}{\sqrt{3^2 - (5x)^2}}\ \mathrm{d}x$$

$$Let : 5x = 3\sin(t)$$

$$5\ \mathrm{d}x = 3\cos(t)\ \mathrm{d}t$$

$$\mathrm{d}x = \frac{3}{5}\cos(t)\ \mathrm{d}t$$

$$\int \frac{(\frac{3}{5}\sin(t))^2}{\sqrt{3^2 - 3^2\sin^2(t)}}\frac{3}{5}\cos(t)\ \mathrm{d}t$$

$$\frac{\frac{27}{125}}{3}\int \frac{\cos(t)\sin^2(t)}{\cos(t)}\ \mathrm{d}t$$

$$\frac{9}{125}\int \sin^2(t)\ \mathrm{d}t$$

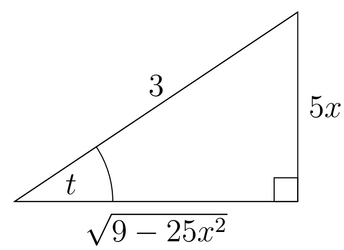
$$\frac{9}{125}\int \frac{1}{2}(1 - \cos(2t))\ \mathrm{d}t$$

$$\frac{9}{250}\left[t - \frac{\sin(2t)}{2}\right] + C$$

$$\frac{9}{250}\left[t - \frac{2\sin(t)\cos(t)}{2}\right] + C$$

$$\frac{9}{250}\left[t - \sin(t)\cos(t)\right] + C$$

Recall that we substituted $5x = 3\sin(t)$, which we can rewrite as $\sin(t) = \frac{5x}{3} = \frac{opp}{hyp}$. If we imagine a triangle in which the opposite side is $5x$ and the hypotenuse is 3, then the adjacent side is $\sqrt{9 - 25x^2}$.



Therefore: $\cos(t) = \frac{adj}{hyp} = \frac{\sqrt{9-25x^2}}{3}$ and $t = \sin^{-1}(\frac{5x}{3})$

$$\frac{9}{250}\left[t - \sin(t)\cos(t)\right] + C$$

$$\frac{9}{250}\left[\sin^{-1}(\frac{5x}{3}) - \frac{5x}{3}\frac{\sqrt{9-25x^2}}{3}\right] + C$$

$$\frac{9}{250}\left[\sin^{-1}(\frac{5x}{3}) - \frac{5x\sqrt{9-25x^2}}{9}\right] + C$$

Practice Problem 27

$$\int \sqrt{x^2 + 2x}\ \mathrm{d}x$$

This problem requires a different approach. We need to turn this into the form of a trigonometric substitution problem.

$$\int \sqrt{x^2 + 2x}\ \mathrm{d}x = \int \sqrt{x^2 + 2x + (1-1)}\ \mathrm{d}x$$

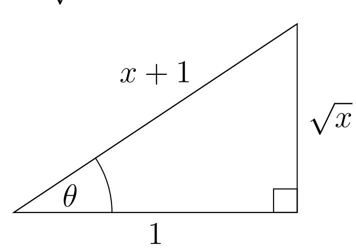
$$\int \sqrt{x^2 + 2x + 1 - 1}\ \mathrm{d}x$$

$$\int \sqrt{(x+1)^2 - 1}\ \mathrm{d}x$$
$$Let : x+1 = \sec(\theta)$$
$$\mathrm{d}x = \sec(\theta)\tan(\theta)\ \mathrm{d}\theta$$
$$\int \sqrt{\sec^2(\theta) - 1}\sec(\theta)\tan(\theta)\ \mathrm{d}\theta$$
$$\int \sqrt{\tan^2(\theta)}\sec(\theta)\tan(\theta)\ \mathrm{d}\theta$$
$$\int \tan^2(\theta)\sec(\theta)\ \mathrm{d}\theta$$

Now we use integration by parts:

$$Let : f(x) = \tan(\theta) \quad g'(x) = \tan(\theta)\sec(\theta)\ \mathrm{d}\theta$$
$$f'(x) = \sec^2(\theta)\ \mathrm{d}\theta \quad g(x) = \sec(\theta)$$
$$\int \tan^2(\theta)\sec(\theta)\ \mathrm{d}\theta = \tan(\theta)\sec(\theta) - \int \sec^2(\theta)\sec(\theta)\ \mathrm{d}\theta$$
$$\int \tan^2(\theta)\sec(\theta)\ \mathrm{d}\theta = \tan(\theta)\sec(\theta) - \int (\tan^2(\theta)+1)\sec(\theta)\ \mathrm{d}\theta$$
$$\int \tan^2(\theta)\sec(\theta)\ \mathrm{d}\theta = \tan(\theta)\sec(\theta) - \int \tan^2(\theta)\sec(\theta)\ \mathrm{d}(\theta) + \int \sec(\theta)\ \mathrm{d}\theta$$
$$2\int \tan^2(\theta)\sec(\theta)\ \mathrm{d}\theta = \tan(\theta)\sec(\theta) - \int \sec(\theta)\ \mathrm{d}\theta$$
$$2\int \tan^2(\theta)\sec(\theta)\ \mathrm{d}\theta = \tan(\theta)\sec(\theta) - \ln|\sec(x)+\tan(x)| + C$$
$$\int \tan^2(\theta)\sec(\theta)\ \mathrm{d}\theta = \frac{1}{2}\tan(\theta)\sec(\theta) - \frac{1}{2}\ln|\sec(x)+\tan(x)| + C$$

Recall that we substituted $x+1 = \sec(\theta)$, which we can rewrite as $\sec(\theta) = \frac{x+1}{1} = \frac{hyp}{adj}$. If we imagine a triangle in which the hypotenuse is $x+1$ and the adjacent side is 1, then the opposite side must be $\sqrt{x}$.



Therefore: $\tan(\theta) = \frac{opp}{adj} = \frac{\sqrt{x}}{1}$

$$\frac{1}{2}\tan(\theta)\sec(\theta) - \frac{1}{2}\ln|\sec(x)+\tan(x)| + C = \frac{1}{2}\sqrt{x}(x+1) - \frac{1}{2}\ln|(x+1)+\sqrt{x}| + C$$

$$= \frac{\sqrt{x}(x+1)}{2} - \frac{\ln|x+\sqrt{x}+1|}{2} + C$$

Practice Problem 45

$$\int x^3\sqrt{1+x^2}\ \mathrm{d}x$$

For this problem, we can solve it with regular substitution:

$$\int x^2 x\sqrt{1+x^2}\ \mathrm{d}x$$

$$Let : 1+x^2 = t$$

$$x\ \mathrm{d}x = \frac{\mathrm{d}t}{2}$$

$$\frac{1}{2}\int (t-1)t^{\frac{1}{2}}\ \mathrm{d}t$$

$$\frac{1}{2}\int t^{\frac{3}{2}} - t^{\frac{1}{2}}\ \mathrm{d}t$$

$$= \frac{1}{2}\left[\frac{t^{\frac{5}{2}}}{\frac{5}{2}} - \frac{t^{\frac{3}{2}}}{\frac{2}{2}} + C\right]$$

But we can also use trigonometric substitution:

$$\int x^3\sqrt{1+x^2}\ \mathrm{d}x$$

$$Let : x = \tan(\theta)$$

$$\mathrm{d}x = \sec^2(\theta)\ \mathrm{d}\theta$$

$$\int \tan^3(\theta)\sqrt{1+\tan^2(\theta)}\sec^2(\theta)\ \mathrm{d}\theta$$

$$\int \tan^3(\theta)\sqrt{\sec^2(\theta)}\sec^2(\theta)\ \mathrm{d}\theta$$

$$\int \tan^3(\theta)\sec^3(\theta)\ \mathrm{d}\theta$$

$$\int \tan(\theta)\tan^2(\theta)\sec^3(\theta)\ \mathrm{d}\theta$$

$$\int \tan(\theta)(\sec^2 - 1)\sec^3(\theta)\ \mathrm{d}\theta$$

$$\int \tan(\theta)\sec^5(\theta)\ \mathrm{d}\theta - \int \tan(\theta)\sec^3(\theta)\ \mathrm{d}\theta$$

$$\int \tan(\theta)\sec(\theta)\sec^4(\theta)\ \mathrm{d}\theta - \int \tan(\theta)\sec(\theta)\sec^2(\theta)\ \mathrm{d}\theta$$

$$Let : u = \sec(\theta)$$

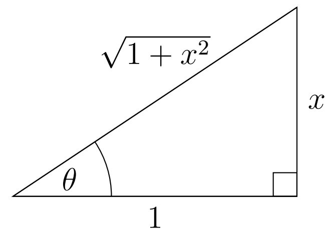
$$\mathrm{d}u = \sec(\theta)\tan(\theta)\ \mathrm{d}\theta$$

$$\int u^4\ \mathrm{d}u - \int u^2\ \mathrm{d}u$$

$$\frac{u^5}{5} - \frac{u^3}{3} + C$$

$$\frac{\sec^5(\theta)}{5} - \frac{\sec^3(\theta)}{3} + C$$

Recall that we substituted $x = \tan(\theta)$, which we can rewrite as $\tan(\theta) = \frac{x}{1} = \frac{opp}{adj}$. If we imagine a triangle in which the opposite side is x and the adjacent side is 1, then the hypotenuse must be $\sqrt{1+x^2}$.



Therefore: $sec(\theta) = \frac{hyp}{adj} = \frac{\sqrt{1+x^2}}{1}$.

$$\frac{\sec^5(\theta)}{5} - \frac{\sec^3(\theta)}{3} + C = \frac{1}{5}(\sqrt{1+x^2})^5 - \frac{1}{3}(\sqrt{1+x^2})^3 + C$$

$$= \frac{(1+x^2)^{\frac{5}{2}}}{5} - \frac{(1+x^2)^{\frac{3}{2}}}{3} + C$$

Follow-up Questions

$$\int x^2\sqrt{3-2x-x^2}\ \mathrm{d}x$$

$$\int x^2\sqrt{-(x^2+2x-3)}\ \mathrm{d}x$$

$$\int x^2\sqrt{-(x^2+2x-3+4-4)}\ \mathrm{d}x$$

$$\int x^2\sqrt{-(x^2+2x+1)+4}\ \mathrm{d}x$$

$$\int x^2\sqrt{4-(x+1)^2}\ \mathrm{d}x$$

$$Let: x+1 = 2\cos(\theta)$$

$$\mathrm{d}x = -2\sin(\theta)\ \mathrm{d}\theta$$

$$\int (2\cos(\theta)-1)^2\sqrt{4-(2\cos(\theta))^2}(-2\sin(\theta))\ \mathrm{d}\theta$$

$$-2\int (2\cos(\theta)-1)^2\sqrt{4}\sqrt{1-cos^2(\theta)}(-2\sin(\theta))\ \mathrm{d}\theta$$

$$-4\int (2\cos(\theta)-1)^2\sqrt{\sin^2(\theta)}(-\sin(\theta))\ \mathrm{d}\theta$$

$$-4\int (4\cos^2(\theta)-4\cos(\theta)+1)(-\sin^2(\theta))\ \mathrm{d}\theta$$

$$4\int 4\cos^2(\theta)\sin^2(\theta)-4\cos(\theta)\sin^2(\theta)+\sin^2(\theta)\ \mathrm{d}\theta$$

$$16\int \cos^2(\theta)\sin^2(\theta)\ \mathrm{d}\theta - 16\int \cos(\theta)\sin^2(\theta)\ \mathrm{d}\theta - 4\int \sin^2(\theta)\ \mathrm{d}\theta$$

$$4\int (2\cos^2(\theta)\sin^2(\theta))^2\ \mathrm{d}\theta - 16\int \cos(\theta)\sin^2(\theta)\ \mathrm{d}\theta - 2\int 2\sin^2(\theta)\ \mathrm{d}\theta$$

$$4\int \sin^2(2\theta)\ \mathrm{d}\theta - 16\int \cos(\theta)\sin^2(\theta)\ \mathrm{d}\theta - 2\int 1-\cos(2\theta)\ \mathrm{d}\theta$$

$$\left[2\int 2\sin^2(2\theta)\ \mathrm{d}\theta\right] - \left[16\frac{\cos^3(\theta)}{3}\right] - \Big[(2\theta-\sin(2\theta))\Big] + C$$

$$\left[\int 1-\cos(4\theta)\ \mathrm{d}\theta\right] - \left[16\frac{\cos^3(\theta)}{3}\right] - \Big[(2\theta-\sin(2\theta))\Big] + C$$

$$\left[2\theta - \frac{\sin(4\theta)}{2}\right] - \left[16\frac{\cos^3(\theta)}{3}\right] - \left[(2\theta - \sin(2\theta))\right] + C$$

$$2\theta - \frac{\sin(4\theta)}{2} - \frac{16\cos^3(\theta)}{3} - 2\theta + \sin(2\theta) + C$$

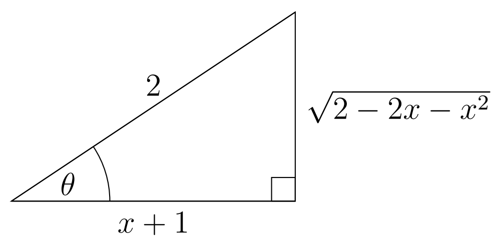
$$\sin(2\theta) - \frac{\sin(4\theta)}{2} - \frac{16\cos^3(\theta)}{3} + C$$

$$2\sin(\theta)\cos(\theta) - \frac{2\sin(2\theta)\cos(2\theta)}{2} - \frac{16\cos^3(\theta)}{3} + C$$

$$2\sin(\theta)\cos(\theta) - \frac{2(2\sin(\theta)\cos(\theta))(2\cos^2(\theta)-1)}{2} - \frac{16\cos^3(\theta)}{3} + C$$

$$2\sin(\theta)\cos(\theta) - (2\sin(\theta)\cos(\theta))(2\cos^2(\theta)-1) - \frac{16\cos^3(\theta)}{3} + C$$

Recall that we substituted $x+1 = 2\cos(\theta)$, which we can rewrite as $\cos(\theta) = \frac{x+1}{2} = \frac{adj}{hyp}$. If we imagine a triangle in which the adjacent side is $x+1$ and the hypotenuse is 2, then the opposide side must be $\sqrt{4-(x+1)^2}$, or $\sqrt{2-2x-x^2}$.



Therefore: $\sin(\theta) = \frac{\sqrt{2-2x-x^2}}{2}$

$$2\sin(\theta)\cos(\theta) - (2\sin(\theta)\cos(\theta))(2\cos^2(\theta)-1) - \frac{16\cos^3(\theta)}{3} + C$$

$$2\frac{\sqrt{2-2x-x^2}}{2}\frac{x+1}{2} - (2\frac{\sqrt{2-2x-x^2}}{2}\frac{x+1}{2})(2(\frac{x+1}{2})^2-1) - \frac{16(\frac{x+1}{2})^3}{3} + C$$

$$\frac{(x+1)\sqrt{2-2x-x^2}}{2} - \frac{(x+1)\sqrt{2-2x-x^2}}{2}(\frac{(x+1)^2}{2}-1) - \frac{16}{3}(\frac{x+1}{2})^3 + C$$

$$= \frac{-(x+1)^3\sqrt{2-2x-x^2}}{2} - \frac{2(x+1)^3}{3} + C$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech