

Trigonometric Integrals

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Trigonometric Integrals

Using concepts from Trigonometric Substitution, we can solve integrals with trigonometric functions by separating out even powers and using the following identities:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

If we can isolate an even power of \sin , we can substitute:

$$\sin^2(\theta) = 1 - \cos^2(\theta)$$

If we can isolate an even power of \cos , we can substitute:

$$\cos^2(\theta) = 1 - \sin^2(\theta)$$

The same rules apply for even powers of \tan , \sec , \cot , and \csc .

Practice Problem 2

$$\int \sin^3(\theta) \cos^4(\theta) \, d\theta$$
$$\int \sin^2(\theta) \sin(\theta) \cos^3(\theta) \, d\theta$$

$$\text{Let : } x = \cos(\theta)$$

$$dx = -\sin(\theta) \, d\theta$$

$$\begin{aligned}
& - \int (1 - \cos^2(\theta)) \cos^4(\theta) \, dx \\
& - \int (1 - x^2)x^4 \, dx \\
& - \int x^4 - x^6 \, dx \\
& -\frac{1}{5}x^5 + \frac{1}{7}x^7 + C \\
& = -\frac{1}{5} \cos^5(\theta) + \frac{1}{7} \cos^7(\theta) + C
\end{aligned}$$

Practice Problem 4

$$\begin{aligned}
& \int_0^{\pi/2} \sin^5(x) \, dx \\
& \int_0^{\pi/2} \sin^4(x) \sin(x) \, dx \\
& \int_0^{\pi/2} (1 - \cos^2(x))^2 \sin(x) \, dx \\
& \int_0^{\pi/2} (\cos^4(x) - 2\cos^2(x) + 1) \sin(x) \, dx
\end{aligned}$$

$$\text{Let } t = \cos(x)$$

$$dt = -\sin(x) \, dx$$

The limits from 0 to $\pi/2$ become 1 to 0. $\left(\int_0^{\pi/2} \rightarrow \int_1^0 \right)$

$$- \int_1^0 t^4 - 2t^2 + 1 \, dt$$

We can flip the integral and negate it to remove the minus sign.

$$\int_0^1 t^4 - 2t^2 + 1 \, dt$$

$$\left[\frac{1}{5}t^5 - \frac{2}{3}t^3 + t \right]_0^1$$

$$\begin{aligned}\frac{1}{5} - \frac{2}{3} + 1 &= \frac{3 - 10 + 15}{15} \\ &= \frac{8}{15}\end{aligned}$$

Practice Problem 13

$$\begin{aligned}&\int \sqrt{\cos(\theta)} \sin^3(\theta) \, d\theta \\ &\int \sqrt{\cos(\theta)} (1 - \sin^2(\theta)) \sin(\theta) \, d\theta \\ &\quad \text{Let : } t = \cos(\theta) \\ &\quad dt = -\sin(\theta) \, d\theta \\ &\quad - \int t^{\frac{1}{2}} (1 - t^2) \, dt \\ &\quad \int -t^{\frac{1}{2}} + t^{\frac{5}{2}} \, dt \\ &\quad -\frac{2}{3} t^{\frac{3}{2}} + \frac{2}{7} t^{\frac{7}{2}} + C \\ &-\frac{2}{3} \cos^{\frac{3}{2}}(\theta) + \frac{2}{7} \cos^{\frac{7}{2}}(\theta) + C\end{aligned}$$

Practice Problem 21

$$\begin{aligned}&\int \tan(x) \sec^3(x) \, dx \\ &\int \tan(x) \sec(x) \sec^2(x) \, dx \\ &\quad \text{Let : } t = \sec(x) \\ &\quad dt = \sec(x) \tan(x) \, dx \\ &\quad \int t^2 \, dt \\ &\quad \frac{1}{3} t^3 + C \\ &= \frac{1}{3} \sec^3(x) + C\end{aligned}$$

Practice Problem 22

$$\int \tan^2(\theta) \sec^4(\theta) \, d\theta$$
$$\int \tan^2(\theta) \sec^2(\theta) \sec^2(\theta) \, d\theta$$
$$\int \tan^2(\theta)(1 + \tan^2(\theta)) \sec^2(\theta) \, d\theta$$

$$\text{Let : } t = \tan(\theta)$$

$$dt = \sec^2(\theta)$$

$$\int t^2(1 + t^2) \, dt$$

$$\int t^2 + t^6 \, dt$$

$$\frac{1}{3}t^3 + \frac{1}{5}t^5 + C$$

$$\frac{1}{3} \tan^3(\theta) + \frac{1}{5} \tan^5(\theta) + C$$

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