Integration By Parts

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Calculus II: August 2016 - December 2016

Integration By Parts

Product Formula:

$$\frac{\mathrm{d}}{\mathrm{d}x}(fg) = fg' + f'g$$

Integrate on both sides:

$$fg = \int fg' \, \mathrm{d}x + \int f'g \, \mathrm{d}x$$

$$\int fg' \, \mathrm{d}x = fg - \int f'g \, \mathrm{d}x$$

Or with uv as variables:

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u$$

For difficult integrals, this method can be used to rewrite the integral into an friendlier form.

Practice problem 1

$$\int x \cos(5x) \, \mathrm{d}x$$

$$Let : f(x) = x \quad g'(x) = \cos(5x) \, \mathrm{d}x$$

$$f'(x) = \, \mathrm{d}x \quad g(x) = \frac{\sin(5x)}{5}$$

$$\int fg' \, \mathrm{d}x = fg - \int f'g \, \mathrm{d}x$$

$$\int x\cos(5x)\ \mathrm{d}x = \frac{x\sin(5x)}{5} - \frac{1}{5}\int \sin(5x)\ \mathrm{d}x$$

$$= \frac{x\sin(5x)}{5} - \frac{1}{5}(-\frac{\cos(5x)}{5}) + C$$

$$= \frac{1}{5}x\sin(5x) + \frac{1}{25}\cos(5x) + C$$

Practice problem 10

$$\int \ln(\sqrt{x})\ \mathrm{d}x$$

$$\frac{1}{2}\int \ln(x)\ \mathrm{d}x = \frac{1}{2}\int 1\ln(x)\ \mathrm{d}x$$

$$Let : g'(x) = 1\ \mathrm{d}x \quad f(x) = \ln(x)$$

$$g(x) = x \quad f'(x) = \frac{1}{x}\ \mathrm{d}x$$

$$\int \ln(\sqrt{x})\ \mathrm{d}x = \frac{1}{2}\left(x\ln(x) - \int x\frac{1}{x}\ \mathrm{d}x\right)$$

$$\frac{1}{2}\left(x\ln(x) - \int 1\ \mathrm{d}x\right)$$

$$= \frac{x\ln(x) - x}{2} + C$$

Practice problem 12

$$\int \tan^{-1} 2u\ \mathrm{d}u$$

$$\int \tan^{-1} 2u\ \mathrm{d}u = \int 1\tan^{-1} 2u\ \mathrm{d}u$$

$$Let : g'(u) = 1\ \mathrm{d}u \quad f(u) = \tan^{-1} 2u$$

$$g(u) = u \quad f'(u) = 2\frac{1}{1+4u^2}\ \mathrm{d}u$$

$$\int \tan^{-1} 2u\ \mathrm{d}u = \left(u\tan^{-1} 2u\right) - \int \frac{2u}{1+4u^2}\ \mathrm{d}u$$

After using integration by parts, we can solve the new integral with u-substitution.

$$\int \frac{2u}{1+4u^2}\ \mathrm{d}u$$

$$Let : x = 1 + 4u^2$$

$$dx = 8u\ \mathrm{d}u$$

$$\int \frac{2u}{x}\frac{\mathrm{d}x}{8u}$$

$$\frac{1}{4}\int \frac{\mathrm{d}x}{x}$$

$$\frac{1}{4}\ln(x) + C$$

$$= \frac{1}{4}\ln(1+4u^2) + C$$

Therefore:

$$\int \tan^{-1} 2u\ \mathrm{d}u$$

$$= u\tan^{-1} 2u - \frac{1}{4}\ln(1+4u^2) + C$$

Practice problem 32

$$\int_1^2 \frac{(\ln(x))^2}{x^3}\ \mathrm{d}x$$

$$Let : u = \ln(x)$$

$$\mathrm{e}^u = x$$

The limit of the integral from 1 to 2 changes to e^1 to e^2

$$\mathrm{e}^u\ \mathrm{d}u =\ \mathrm{d}x$$

$$\int_{\mathrm{e}}^{\mathrm{e}^2} \frac{u^2\mathrm{e}^u\ \mathrm{d}u}{(\mathrm{e}^u)^3}$$

$$= \int_{\mathrm{e}}^{\mathrm{e}^2} u^2\mathrm{e}^{-2u}\ \mathrm{d}u$$

Now we can use integration by parts:

$$Let : f(x) = u^2 \quad g'(x) = \mathrm{e}^{-2u}$$

$$f'(x) = 2u \text{ d}u \quad g(x) = \frac{\mathrm{e}^{-2u}}{-2}$$

$$\int_{\mathrm{e}}^{\mathrm{e}^2} u^2\mathrm{e}^{-2u} \text{ d}u = \left(u^2\frac{\mathrm{e}^{-2u}}{-2}\right) - \int 2u\frac{\mathrm{e}^{-2u}}{-2} \text{ d}u$$

$$= \frac{u^2\mathrm{e}^{-2u}}{-2} + \int u\mathrm{e}^{-2u} \text{ d}u$$

In order to solve the integral properly, you need to use integration by parts again:

$$Let : f(x) = u \quad g'(x) = \mathrm{e}^{-2u} \text{ d}u$$

$$f'(x) = 1 \text{ d}u \quad g(x) = \frac{\mathrm{e}^{-2u}}{-2}$$

$$\frac{u^2\mathrm{e}^{-2u}}{-2} + \left(\frac{u^2\mathrm{e}^{-2u}}{-2} - \int (1)\frac{\mathrm{e}^{-2u}}{-2} \text{ d}u\right)$$

$$\frac{u^2\mathrm{e}^{-2u}}{-2} + \left(\frac{u^2\mathrm{e}^{-2u}}{-2} + \frac{1}{2}\int \mathrm{e}^{-2u} \text{ d}u\right)$$

$$-u^2\mathrm{e}^{-2u} + \left(\frac{1}{2}\int \mathrm{e}^{-2u} \text{ d}u\right)$$

$$\left[-u^2\mathrm{e}^{-2u} + \frac{1}{2}(\frac{e^{-2u}}{-2})\right]_{\mathrm{e}}^{\mathrm{e}^2}$$

$$-\mathrm{e}^4\mathrm{e}^{-2e^2} - \frac{\mathrm{e}^{-2\mathrm{e}^2}}{4} - \left[-\mathrm{e}^2\mathrm{e}^{-2e} - (\frac{e^{-2\mathrm{e}}}{4})\right]$$

$$-\mathrm{e}^4\mathrm{e}^{-2e^2} - \frac{\mathrm{e}^{-2\mathrm{e}^2}}{4} + \mathrm{e}^2\mathrm{e}^{-2e} + \frac{e^{-2\mathrm{e}}}{4}$$

$$= -\mathrm{e}^{-8\mathrm{e}^2} + \mathrm{e}^{-4\mathrm{e}} + \frac{\mathrm{e}^{-2\mathrm{e}^2+\mathrm{e}^{-2\mathrm{e}}}}{4}$$

Practice problem 38

$$\int \cos(\ln(x))\ \mathrm{d}x$$

$$Let : u = \ln(x)$$

$$x = \mathrm{e}^u$$

$$\mathrm{d}x = \mathrm{e}^u\ \mathrm{d}u$$

$$\int \cos(u)\mathrm{e}^u\ \mathrm{d}u = \int \mathrm{e}^u \cos(u)\ \mathrm{d}u$$

We can describe this integral in a more general form as:

$$\mathrm{I} = \int \mathrm{e}^{ax} \cos(bx)\ \mathrm{d}x$$

where a and b are both 1. Using integration by parts where:

$$f(x) = \mathrm{e}^{ax} \quad g'(x) = \cos(bx)\ \mathrm{d}(x)$$

we can get:

$$\frac{\mathrm{e}^{ax}\sin(bx)}{b} - \int \frac{a\mathrm{e}^{ax}\sin(bx)}{b}\ \mathrm{d}x$$

$$\frac{\mathrm{e}^{ax}\sin(bx)}{b} - \frac{a}{b}\int \mathrm{e}^{ax}\sin(bx)\ \mathrm{d}x$$

Using integration by parts again:

$$\frac{\mathrm{e}^{ax}\sin(bx)}{b} - \frac{a}{b}\left[\mathrm{e}^{ax}(\frac{-\cos(bx)}{b}) - \int a\mathrm{e}^{ax}(\frac{-\cos(bx)}{b})\ \mathrm{d}x\right]$$

$$\mathrm{I} = \int \mathrm{e}^{ax}\cos(bx)\ \mathrm{d}x = \frac{\mathrm{e}^{ax}\sin(bx)}{b} + \frac{a}{b^2}\mathrm{e}^{ax}\cos(bx) - \frac{a^2}{b^2}\int \mathrm{e}^{ax}\cos(bx)\ \mathrm{d}x$$

Rewriting this:

$$\int \mathrm{e}^{ax}\cos(bx)\ \mathrm{d}x + \frac{a^2}{b^2}\int \mathrm{e}^{ax}\cos(bx)\ \mathrm{d}x = \frac{\mathrm{e}^{ax}}{b^2}[b\sin(bx) + a\cos(bx)]$$

$$= \left(1 + \frac{a^2}{b^2}\right)\int \mathrm{e}^{ax}\cos(bx)\ \mathrm{d}x = \frac{\mathrm{e}^{ax}}{b^2}[b\sin(bx) + a\cos(bx)]$$

$$(b^2 + a^2)\int \mathrm{e}^{ax}\cos(bx)\ \mathrm{d}x = \mathrm{e}^{ax}[b\sin(bx) + a\cos(bx)]$$

$$\mathrm{I} = \int \mathrm{e}^{ax}\cos(bx)\ \mathrm{d}x = \frac{\mathrm{e}^{ax}}{a^2+b^2}[b\sin(bx)+a\cos(bx)]$$

We can apply this general form back to the original integral.

$$\int \mathrm{e}^{u}\cos(u)\ \mathrm{d}u$$

For this case:

$$a = 1 \quad b = 1$$

$$\int \mathrm{e}^{u}\cos(u)\ \mathrm{d}u = \frac{\mathrm{e}^{x}}{1+1}[\sin(x)+\cos(x)]$$

$$= \frac{\mathrm{e}^{x}(\sin(x)+\cos(x))}{2}$$

Practice Problem 39

$$\int \theta^3\cos(\theta^2)\ \mathrm{d}\theta$$

First we start with subsitution:

$$Let : \theta^2 = x$$

$$2\theta = \frac{\mathrm{d}x}{\mathrm{d}\theta} \quad \frac{\mathrm{d}x}{2} = \theta\ \mathrm{d}\theta$$

$$\int \theta^2\cos(\theta^2)\theta\ \mathrm{d}\theta = \frac{1}{2}\int x\cos(x)\ \mathrm{d}x$$

Then we can use integration by parts:

$$\int x\cos(x)\ \mathrm{d}x$$

$$Let : u = x \quad \mathrm{d}v = \cos(x)\ \mathrm{d}x$$

$$\mathrm{d}u = \ \mathrm{d}x \quad v = \sin(x)$$

$$x\cos(x)\ \mathrm{d}x = x\sin(x) - \int \sin(x)\ \mathrm{d}x$$

$$x\sin(x) - \cos(x) + C$$

Now we can substitute this back:

$$\frac{1}{2}\int x\cos(x)\ \mathrm{d}x = \frac{1}{2}x\sin(x) - \cos(x) + C$$

$$= \frac{1}{2}\theta^2\sin(\theta^2) - \cos(\theta^2) + C$$

Practice Problem 44

$$\int x^{\frac{3}{2}} \ln(x)\ \mathrm{d}x$$

$$Let : u = \ln(x) \quad \mathrm{d}v = x^{\frac{3}{2}}\ \mathrm{d}x$$

$$\mathrm{d}u = \frac{1}{x}\ \mathrm{d}x \quad v = \frac{x^{\frac{5}{2}}}{\frac{5}{2}}$$

$$\int x^{\frac{3}{2}} \ln(x)\ \mathrm{d}x = \frac{x^{\frac{5}{2}} \ln(x)}{\frac{5}{2}} - \int \frac{1}{x} \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\ \mathrm{d}x$$

$$\frac{2}{5} \ln(x) x^{\frac{5}{2}} - \frac{2}{5} \int x^{\frac{3}{2}}\ \mathrm{d}x$$

$$\frac{2}{5} \ln(x) x^{\frac{5}{2}} - \frac{2}{5} \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\ \mathrm{d}x$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech