Volumes By Integration (Disks)

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Calculus II: August 2016 - December 2016

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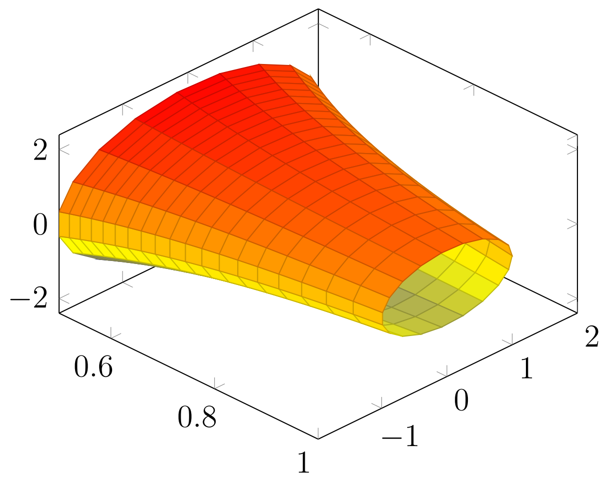


We can take the volume of the above solid by taking areas of cross-sectional slices (A_i) of the volume.

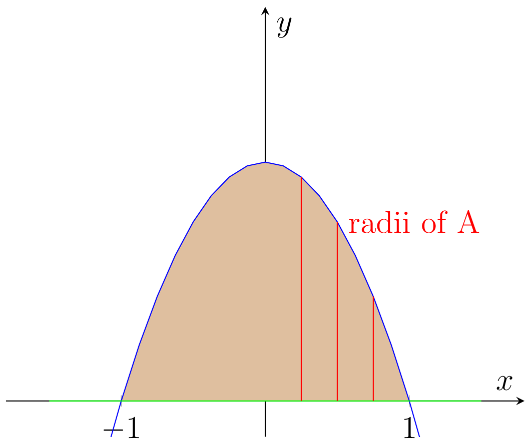
$$V_i = A_i \Delta x$$

$$V \approx \sum_{i=1}^{n} A_i \Delta x \approx \lim\limits_{i=1}^{n} V_i$$

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} V_i = \int_a^b A \, \mathrm{d}x$$

Example 1

Find the volume of $y = 1 - x^2$ rotated about the x-axis.



$$V = \int A \, \mathrm{d}x$$

$$V = \int \pi r^2 \, \mathrm{d}x$$

$$V = \int_{-1}^{1} \pi (1 - x^2)^2 \, \mathrm{d}x$$

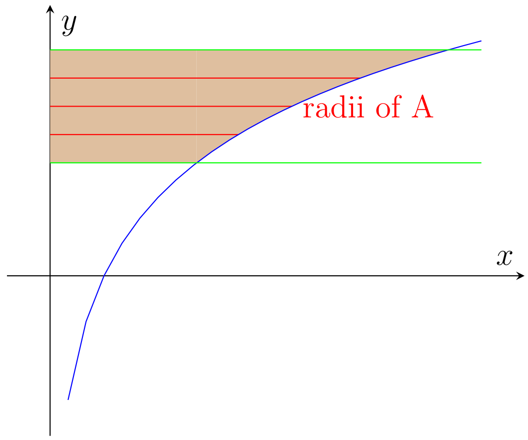
$$V = \pi \int_{-1}^{1} 1 - x^4 - 2x^2 \, \mathrm{d}x$$

$$V = \pi \left[x + \frac{x^5}{5} - \frac{2x^3}{3} \right]_{-1}^{1}$$

$$V = \frac{16\pi}{15}$$

Example 2

Find the volume of $y = \ln(x)$ bounded by the lines $y = 1$, $y = 2$, $x = 0$ when rotated about the y-axis.



$$V = \int A \; \mathrm{d}y$$

$$V = \int \pi r^2 \; \mathrm{d}y$$

$$V = \int_1^2 \pi (x)^2 \; \mathrm{d}y$$

$$V = \int_1^2 \pi (\mathrm{e}^y)^2 \; \mathrm{d}y$$

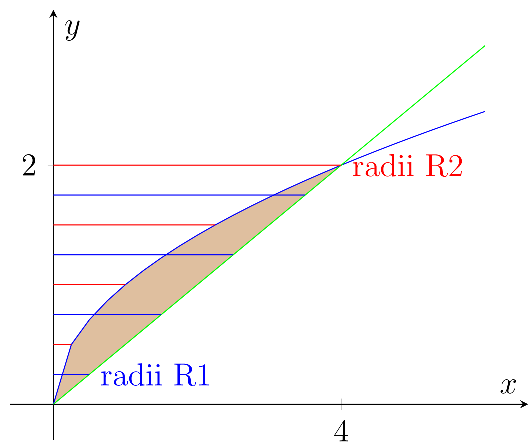
$$V = \pi \int_1^2 \mathrm{e}^{2y} \; \mathrm{d}y$$

$$V = \pi \left[\frac{\mathrm{e}^{2y}}{2} \right]_1^2$$

$$V = \frac{\pi}{2} \left[\mathrm{e}^4 - \mathrm{e}^2 \right]$$

Example 3

Find the volume of the region enclosed between $y^2 = x$ and $x = 2y$ when it is rotated about the y-axis.



$$V = \int A \; \mathrm{d}y$$

$$V = \int \pi (R1)^2 - (R2)^2 \; \mathrm{d}y$$

$$V = \pi \int_0^2 (2y)^2 - (y^2)^2 \; \mathrm{d}y$$

$$V = \pi \int_0^2 4y^2 - y^4 \; \mathrm{d}y$$

$$V = \pi \left[\frac{4y^3}{3} - \frac{y^5}{5} \right]_0^2$$

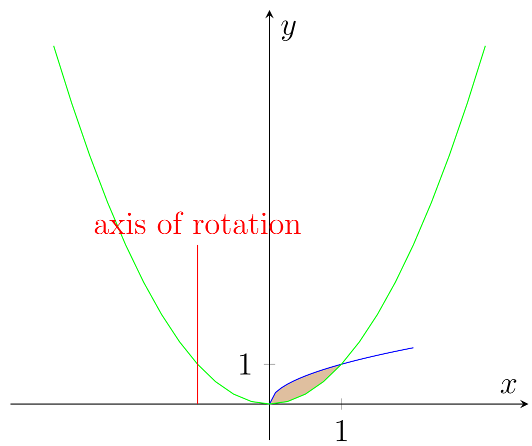
$$V = \frac{32\pi}{3} - \frac{32\pi}{5}$$

Example 4

$$y = x^2 \quad x = y^2$$

rotated about:

$$x = -1$$



$$V = \int A \ \mathrm{d}y$$

$$V = \int \pi (R1)^2 - (R2)^2 \ \mathrm{d}y$$

$$V = \pi \int_0^1 [(1+x)^2 - (1+x)^2] \ \mathrm{d}y$$

$$V = \pi \int_0^1 [(1+\sqrt{y})^2 - (1+y^2)^2] \ \mathrm{d}y$$

$$V = \pi \int_0^1 [1 + y + 2\sqrt{y} - 1 - y^4 - 2y^2] \ \mathrm{d}y$$

$$V = \pi \left[y + \frac{y^2}{2} + \frac{4y^{3/2}}{3} - y - \frac{y^5}{5} - \frac{2y^3}{3} \right]$$

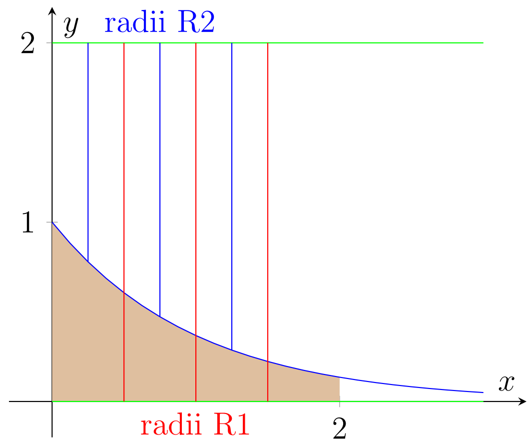
$$V = \frac{29\pi}{30}$$

Practice Problem 12

$$y = \mathrm{e}^{-x} \quad y = 1 \quad x = 2$$

rotated about:

$$y = 2$$



$$V = \int A \, \mathrm{d}x$$

$$V = \int \pi (R1)^2 - (R2)^2 \, \mathrm{d}y$$

$$V = \pi \int_0^2 [(2-y)^2 - 1^2] \, \mathrm{d}x$$

$$V = \pi \int_0^2 [(2-\mathrm{e}^{-x})^2 - 1] \, \mathrm{d}x$$

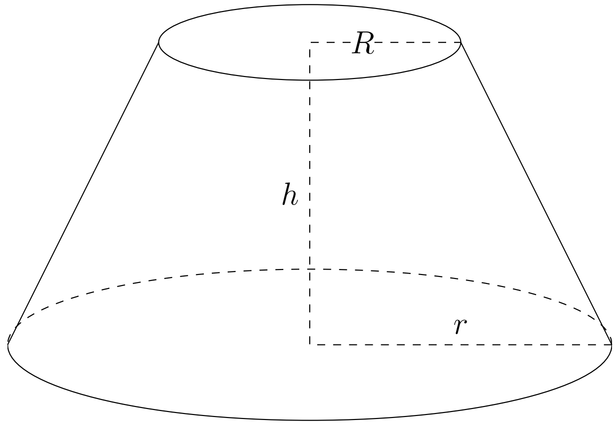
$$V = \pi \int_0^2 [4 - \mathrm{e}^{-2x} - 4\mathrm{e}^{-x} - 1] \, \mathrm{d}x$$

$$V = \pi \int_0^2 [3 + \mathrm{e}^{2x} - 4\mathrm{e}^{-x}] \, \mathrm{d}x$$

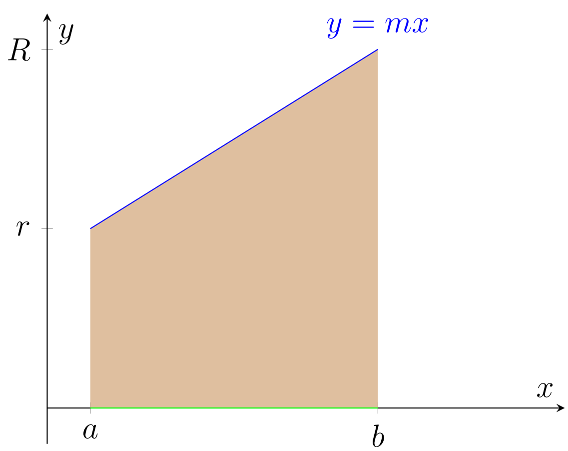
$$V = \pi \left[3x + \frac{\mathrm{e}^{-2x}}{-2} + 4\mathrm{e}^{-x} \right]_0^2$$

$$V = \pi \left[(6 - \frac{1}{2}\mathrm{e}^{-4} + 4\mathrm{e}^{-2}) - (\frac{-1}{2} + 4) \right]$$

Volume of a Frustum



We can turn this problem into an integration problem by taking a slice of the frustum and revolving it around the x-axis.



$$a - b = h \quad y = mx$$
$$r = ma \quad R = mb$$
$$V = \int A \, \mathrm{d}x$$
$$V = \int \pi r^2 \, \mathrm{d}x$$

$$V = \pi \int_a^b (y)^2 \, \mathrm{d}x$$

$$V = \pi \int_a^b (mx)^2 \, \mathrm{d}x$$

$$V = \pi(m^2)\left[\frac{x^3}{3}\right]_a^b$$

$$V = \pi(m^2)(\frac{b^3}{3} - \frac{a^3}{3})$$

$$V = \frac{\pi(m^2)}{3}(b^3 - a^3)$$

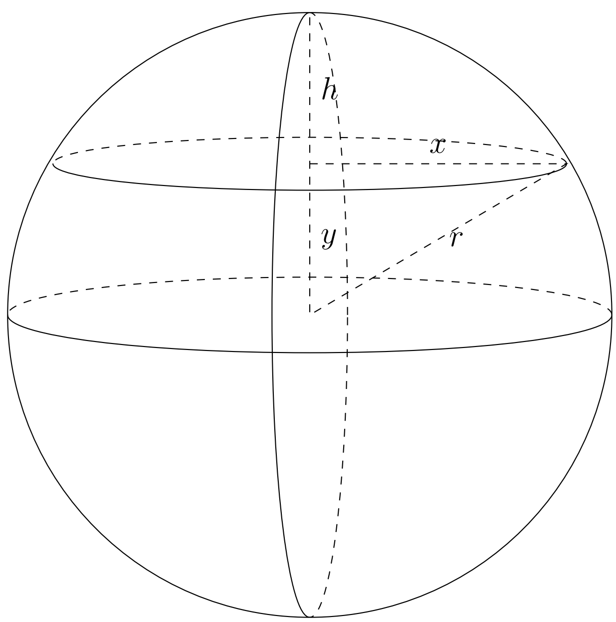
$$V = \frac{\pi(m^2)}{3}(b - a)(b^2 + ab + a^2)$$

Since we know $b - a = h$, $r = ma$, and $R = mb$:

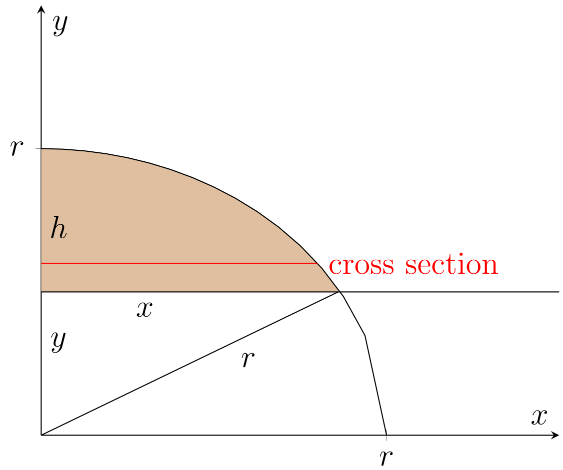
$$V = \frac{\pi(m^2)h}{3}((\frac{r}{m})^2 + \frac{Rr}{m^2} + (\frac{R}{m})^2)$$

$$V = \frac{\pi h}{3}[r^2 + R^2 + Rr]$$

Volume of a Section of a Sphere



The same proof as above can be applied to most shapes. For example, a section of a sphere:



From the Pythagorean theorem, we know that $x^2 + y^2 = r^2$, thus it follows that $x^2 = r^2 - y^2$.

$$V = \int A \; \mathrm{d}y$$

$$V = \int \pi x^2 \; \mathrm{d}y$$

$$V = \int_{r-h}^{r} \pi(r^2 - y^2) \; \mathrm{d}y$$

$$V = \pi \left[r^2 y - \frac{y^3}{3} \right]_{r-h}^{r}$$

$$V = \pi(h^2 r - \frac{h^3}{3})$$

$$V = \pi h^2 (r - \frac{1}{3}h)$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech