

Probability and Statistics

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Expected Values and Covariance

Let X and Y be discrete random variables with joint pmf $p(x, y)$. The expected value of $h(X, Y)$ is:

$$\begin{aligned} E[h(X, Y)] &= \sum_x \sum_y h(x, y)p(x, y) \\ &= \sum_x \sum_y h(x, y)P(X = x \text{ and } Y = y) \\ \mu_x &= \sum_x xp_x(x) \\ \mu_y &= \sum_y yp_y(y) \\ &= \sum_y yP(Y = y) \end{aligned}$$

$p_x(x)$ and $p_y(y)$ are marginal probability mass functions of X and Y , respectively. Recall the formal definitions:

$$\begin{aligned}
 p_x(x) &= \sum_y p(x, y) \\
 p_y(y) &= \sum_x p(x, y) \\
 (\sigma_x)^2 &= \sum_x (x - \mu_x)^2 p_x(x) \\
 &= \sum_x (x - \mu_x)^2 P(X = x) \\
 (\sigma_y)^2 &= \sum_y (y - \mu_y)^2 p_y(y) \\
 \sigma_x &= \sqrt{(\sigma_x)^2} \\
 \sigma_y &= \sqrt{(\sigma_y)^2}
 \end{aligned}$$

The **covariance** between the jointly distributed random variables X and Y is:

$$\begin{aligned}
 Cov(X, Y) &= E \left[(x - \mu_x)(y - \mu_y) \right] \\
 &= \sum_x \sum_y (x - \mu_x)(y - \mu_y) p(x, y)
 \end{aligned}$$

Let X and Y be jointly distributed random variables.

$$Cov(X, Y) = E(XY) - \mu_x \mu_y$$

where:

$$E(XY) = \sum_x \sum_y xy p(x, y)$$

Correlation Coefficient

The correlation coefficient of X and Y , denoted by $Corr(X, Y)$, $\rho_{x,y}$, or just ρ , is:

$$\rho_{x,y} = \frac{Cov(X, Y)}{\sigma_x \sigma_y}$$

If the variables a and c are both positive or both negative, then:

$$\text{Corr}(aX + b, cY + d) = \text{Corr}(X, Y)$$

$$-1 \leq \text{Corr}(X, Y) \leq 1$$

If X and Y with joint pmf $p(x, y)$ and joint pdf $f(x, y)$ are independent, the $\rho = 0$. If $\rho = 0$, then the random variables X and Y may or may not be independent. $\rho = \pm 1$ if and only if $Y = aX + b$ for some numbers a and b with $a \neq 0$.

Uses of the Correlation Coefficient

Let X be the random variable for the height of a randomly selected person at RIT and let Y be the random variable for their weight. $\text{Corr}(X, Y)$ is not likely to be 1 or -1 because there is no strong correlation between height and weight.

Let X be the average temperature of a randomly selected city in degrees Centigrade, with Y being the average temperature expressed in degrees Fahrenheit. The correlation coefficient in this case is 1.

Example

Randomly select a card from a file containing three cards:

- (Name, Readiness Assessment Score, Common Core Score)
- (James, 58, 81)
- (Mike, 73, 78)
- (Jane, 76, 85)

Let X be the random variable for the readiness assessment score and Y be the random variable for the Common Core score. Recall that a random variable is a function. Find the correlation coefficient of X and Y .

x	y	$p(x, y)$	$p_x(x)$	$p_y(y)$	$p_x(x)p_y(y)$
58	81	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{9}$
73	78	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{9}$
76	85	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{9}$

Are X and Y independent?

No, $p(x, y) \neq p_x(x)p_y(y)$.

$$\begin{aligned}\mu_x &= \sum_x xp_x(x) \\ &= 58 \cdot p_x(58) + 73 \cdot p_x(73) + 76 \cdot p_x(76) \\ &= 58 \cdot \frac{1}{3} + 73 \cdot \frac{1}{3} + 76 \cdot \frac{1}{3} = 69\end{aligned}$$

$$\begin{aligned}\mu_y &= \sum_y p_y(y) \\ &= 81 \cdot \frac{1}{3} + 78 \cdot \frac{1}{3} + 85 \cdot \frac{1}{3} = \frac{244}{3}\end{aligned}$$

$$\begin{aligned}(\sigma_x)^2 &= \sum_x (x - \mu_x)^2 p_x(x) \\ &= (58 - 69)^2 \cdot \frac{1}{3} + (73 - 69)^2 \cdot \frac{1}{3} + (76 - 69)^2 \cdot \frac{1}{3} \\ &= \frac{1}{3} [11^2 + 4^2 + 7^2] \\ &= 62\end{aligned}$$

$$\begin{aligned}(\sigma_y)^2 &= \sum_y (y - \mu_y)^2 p_y(y) \\ &= \left(81 - \frac{244}{9}\right)^2 \cdot \frac{1}{3} + \left(78 - \frac{244}{9}\right)^2 \cdot \frac{1}{3} + \left(85 - \frac{244}{9}\right)^2 \cdot \frac{1}{3} \\ &= \frac{1}{3} \left[\left(\frac{1}{3}\right)^2 + \left(-\frac{10}{3}\right)^2 + \left(\frac{11}{3}\right)^2\right] \\ &= \frac{74}{9}\end{aligned}$$

$$\begin{aligned}
Cov(X, Y) &= E\left[(X - \mu_x)(Y - \mu_y)\right] \\
&= E(XY) - \mu_x\mu_y \\
&= \left[\sum_x \sum_y xy p(x, y)\right] - \mu_x\mu_y \\
&= 58 \cdot 81 \cdot \frac{1}{3} + 73 \cdot 78 \cdot \frac{1}{3} + 76 \cdot 85 \cdot \frac{1}{3} - 69 \cdot \left(81 + \frac{1}{3}\right) \\
&= 5.33 \\
Corr(X, Y) &= \frac{5.33}{\sqrt{62}\sqrt{8.22}} = 0.236
\end{aligned}$$

The positive correlation indicates that we can reasonable expect there is a correlation between the Readiness Assessment score and the Common Core score. The correlation is not perfectly linear however.

Example

A surveyor wishes to lay out a square region with each side having length L . However, because of measurement error, he instead lays out a rectangle in which both the north-south sides are of length X and the east-west sides are both of length Y . Suppose X and Y are independent and that each is uniformly distributed on the interval $[L - A, L + A]$ where $0 < A < L$. What is the expected area of the resulting rectangle? Find the joint pdf of X and Y .

$$f(x, y) = \begin{cases} h & , L - A \leq x \leq L + A \text{ and } L - A \leq y \leq L + A \\ 0 & , \text{otherwise} \end{cases}$$

How do we find the value of h ?

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy \\
&= \int_{L-A}^{L+A} \int_{L-A}^{L+A} h dx dy \\
&= h \int_{L-A}^{L+A} \int_{L-A}^{L+A} dx dy \\
&= h(2A)(2A) \\
h &= \frac{1}{4A^2}
\end{aligned}$$

The expected value of the area of the rectangle is $E(XY)$.

$$\begin{aligned}
 E(XY) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy p(x, y) dx dy \\
 &= \int_{L-A}^{L+A} \int_{L-A}^{L+A} xy h dx dy \\
 &= h \int_{L-A}^{L+A} y \left[\int_{L-A}^{L+A} x dx \right] dy \\
 &= h \left[\int_{L-A}^{L+A} x dx \right] \left[\int_{L-A}^{L+A} y dy \right] \\
 &= h \left[\int_{L-A}^{L+A} x dx \right]^2 \\
 &= h \left[\frac{1}{2} (x^2)_{x=L-A}^{x=L+A} \right] \\
 &= h \frac{1}{2} \left[(L+A)^2 - (L-A)^2 \right]^2 \\
 &= h \frac{1}{2} \left[((L+A) + (L-A))((L+A) - (L-A)) \right]^2 \\
 &= h \frac{1}{2} \left[2L \cdot 2A \right]^2 \\
 &= L^2
 \end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech