Probability and Statistics

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Probability

The sample space of an experiment, denoted by S, is the set of all possible outcomes of the experiment. An event of the experiment is a subset of S. An event is simple if it consists of exactly one outcome, and compound if it consists of more than one outcome.

Let A and B be events of an experiment. If $A = \varnothing$, then it is said to be the null event.

If $A \cap B = \varnothing$, then A and B are said to be mutually exclusive, or disjoint.

Roll a Die

$$S = \{1, 2, 3, 4, 5, 6\}$$

Sample events:

- E_1 : getting a 1
- E_2 : getting an even number
- E_3 : getting a number greater than 4
- E_4 : getting a number greater than 9
- E_5 : getting an integer greater than 0 and less than 7

$$
\begin{aligned}
E_1 &= \{1\} \subset S \\
E_2 &= \{2, 4, 6, \} \subset S \\
E_3 &= \{5, 6\} \subset S \\
E_4 &= \{\} = \varnothing \subset S \\
E_5 &= \{1, 2, 3, 4, 5, 6\} \subset S
\end{aligned}
$$

E_1 is a simple event. E_2, E_3, E_5 are compound events. E_4 is the null event.

The events E_1 and E_2 are mutually exclusive or disjoint.

$$E_1 \cap E_2 = \varnothing$$

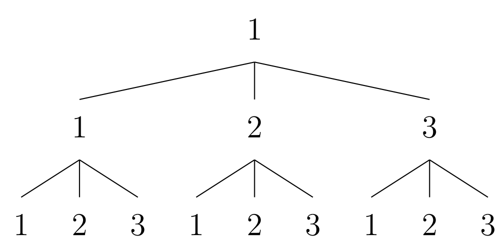
The events E_2 and E_3 are not mutually exclusive or disjoint.

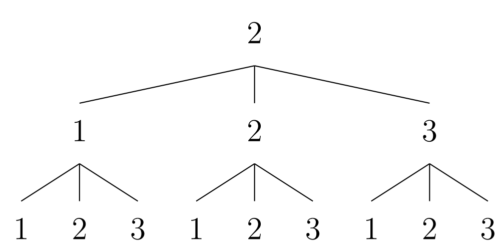
$$E_2 \cap E_3 = \{6\} \neq \varnothing$$

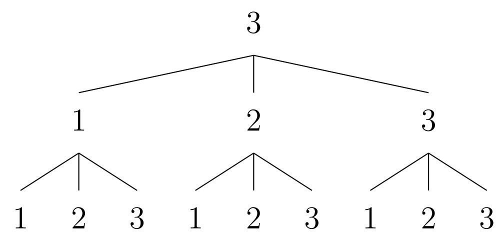
Example

A family consisting of three persons - A, B, and C - goes to a medical clinic that always has a doctor at stations 1, 2, and 3. During a certain week, each member of the family visits the clinic once and is assigned at random to a station. The experiment consists of recording the station number for each member.

A. List 27 outcomes in the sample space.







B. List all outcomes in the event that all three members go to the same station.

$$111, 222, 333$$

C. List all outcomes in the event that all members go to different stations.

$$123, 132, 213, 231, 312, 321$$

There are $3! = 6$ possibilities for these outcomes.

D. List all outcomes in the event that no one goes to station 2.

$$111, 113, 131, 133, 311, 313, 331, 333$$

Example

Consider the experiment of tossing a coin which has U on one side and D on the other side.
Sample space:

$$S = \left\{ U, D \right\}$$

All possible events:

$$\varnothing \quad \left\{ U \right\} \quad \left\{ D \right\} \quad \left\{ U, D \right\}$$

The probability of these events are:

$$\begin{aligned} P(\{\}) &= 0 \\ P(\{U\}) &= \frac{1}{2} \\ P(\{D\}) &= \frac{1}{2} \\ P(\{U, D\}) &= 1 \end{aligned}$$

Let 2^S denote the power set of S. Then:

$$2^S = \Big\{\varnothing, \{U\}, \{D\}, \{U, D\}\Big\}$$

P can be viewed as a function from 2^S to $\mathbb{R}$.

$$P : 2^S \to \mathbb{R}$$

$$\Big\{\varnothing\Big\} \to 0$$

$$\Big\{\{U\}, \{D\}\Big\} \to \frac{1}{2}$$

$$\Big\{\{U, D\}\Big\} \to 1$$

Let:

$$\alpha = \Big\{(S, f : 2^S \to \mathbb{R}) \mid \text{S is a nonempty countable set}\Big\}$$

$$\beta = \Big\{(S, f : 2^S \to \mathbb{R}) \in \alpha \mid (S, f) \text{ satisfies conditions } 1,\ 2,\ \text{and } 3\Big\}$$

We can choose some conditions 1, 2, and 3 to research further about β.

Axioms

A. For any event $A, P(A) \geq 0$

B. $P(S) = 1$

C. If $A_1, A_2, A_3, \dots$ is an infinite collection of disjoint events, the $P(A_1 \cup A_2 \cup A_3 \cup \dots) = \sum_{i=0}^{\infty} P(A_i)$.

$P(\varnothing) = 0$ is proven from the axioms. The proof does not rely on the following definition:

$$P(E) = \frac{\text{\# of outcomes favorable to E}}{\text{total \# of possible outcomes}}$$

Example

Toss a coin and select a ball from a basket containing 3 balls marked R, B, and G.

$$E_1 : \textit{getting a } B$$
$$E_2 : \textit{getting an } H \textit{ and then a } B \textit{ or } G$$
$$E_3 : \textit{not } E_2$$
$$E_4 : \textit{getting an } H \textit{ and then a } B$$

Consider the following probabilities:

$$P(E_1) = \frac{2}{6}$$
$$P(E_2) = \frac{2}{6}$$
$$P(E_3) = \frac{4}{6}$$
$$P(E_4) = \frac{1}{6}$$

E_4 is a simple event.

$$\begin{aligned} P(E_2 \textit{ or } E_3) &= P(E_2 \textit{ or not } E_2) \\ &= P(S) \\ &= 1 \\ &= P(E_2 \cup E_3) \qquad = \\ P(E_2 \textit{ or } E_3) &= P(E_2) + P(E_3) \end{aligned}$$

Example

A 5-sided rock has the following experimental result from tossing 1000 times.

Side	1	2	3	4	5
Frequency	320	180	150	130	220

Our experiment: toss that rock and then toss a coin.

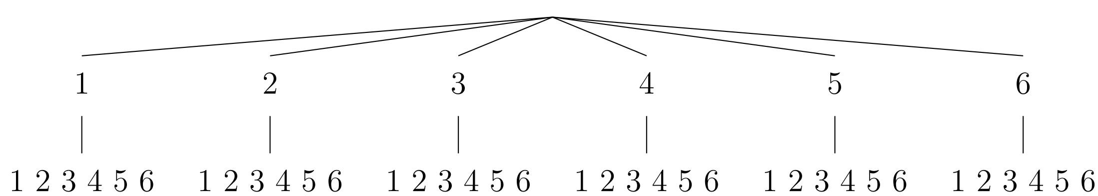
$$E_1 : \textit{a 2 and then an } H$$
$$E_2 : \textit{an odd number and then an } H$$
$$E_3 : \textit{an odd number or 2 and then } H$$

Probabilities:

$$
\begin{aligned}
P(E_1) &\neq \frac{1}{10} \\
P(E_1) &= \frac{180}{1000}\frac{1}{2} \\
P(E_2) &= \frac{320}{1000}\frac{1}{2} + \frac{150}{1000}\frac{1}{2} + \frac{220}{1000}\frac{1}{2} \\
P(E_3) &= P(E_2) + \frac{180}{1000}\frac{1}{2}
\end{aligned}
$$

Example

Roll a die twice.
E: the sum of the two numbers is at least 3



$$P(E) = \frac{\#\ of\ outcomes\ favorable\ to\ E}{total\ \#\ of\ possible\ outcomes}$$

We can sum up the number of rows where the sum is 3, 4, 5, etc, but since the number of outcomes favorable to "not E" is a disjoint set, we can sum up the number of rows where the sum is 2, 1, and 0.
Number of rows where the sum is 2: 1
Number of rows where the sum is 1: 0
Number of rows where the sum is 0: 0

$$P(not\ E) = \frac{1}{36}$$

$$P(E) + P(not\ E) = 1$$

$$\therefore P(E) = 1 - P(not\ E) = 1 - \frac{1}{36} = \frac{35}{36}$$

Example

The manager at Arango Automotive has found that the probability that a car brought into the shop requires an oil change is 0.6, the probability that a car requires brake repair is 0.2, and the probability that a car requires both an oil change and brake repair is 0.1. For a car brought into the shop, determine the probability that the car will require an oil change or brake repair.

$$\begin{aligned} P(oil \cup brake) &= P(oil) + P(brake) - P(oil \cap brake) \\ &= 0.6 + 0.2 - 0.1 \\ &= 0.7 \end{aligned}$$

General formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If you have any questions, comments, or concerns, please contact me at
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