

# Sets

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Discrete Math for Computing: January 2017 - May 2017

## Mathematical Induction

Mathematical inductions prove statements that depend on a positive integer  $n$ . In general, we use it to prove  $\forall n P(n)$  where we normally have  $\mathbb{U} = \mathbb{Z} \geq a$  for  $a = \{0, 1, 2, 3, 4, 5, \dots\}$ . Unless otherwise specified, we assume  $a = 1$ .

### Examples

For all  $n > 0$ :

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

For all  $n > 0$ :

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

For all  $n > 0$ , 2 divides  $n^2 + n$ :

$$n^2 + n \equiv 0 \pmod{2}$$

For all  $n > 0$ :

$$4^{n+1} + 5^{2n-1} \equiv 0 \pmod{21}$$

### Process

1. Identify the exact  $P(n)$  statement.
2. Write out  $P(k)$  in the induction hypothesis.

3. Write out  $P(k + 1)$  in the induction.
4. Think about how  $P(k + 1)$  relates to  $P(k)$ .
5. Once you apply the induction hypothesis in the induction, the rest is algebra.

With the math induction argument, we have:

$$\begin{array}{l} P(1) \\ \hline \forall k P(k) \rightarrow P(k + 1) \\ \hline \therefore \forall n P(n) \end{array}$$

Using Modus Ponens:

$$\begin{array}{l} P(1) \\ P(1) \rightarrow P(2) \\ \hline \therefore P(2) \end{array}$$

By hypothetical syllogism:

$$\begin{array}{l} P(1) \rightarrow P(2) \\ P(2) \rightarrow P(3) \\ \hline \therefore P(1) \rightarrow P(3) \end{array}$$

## Example

Prove  $\forall n \sum_{i=1}^n (2i - 1) = n^2$ .

1. Basis:  $P(n) = \sum_{i=1}^n (2i - 1) = n^2$
2. Show  $P(1)$  is true:

$$\begin{array}{l} \sum_{i=1}^1 (2i - 1) = 1^1 \\ (2 \cdot 1 - 1) = 1^2 \\ 2 - 1 = 1 \end{array}$$

3. Induction hypothesis, assume  $P(k)$  is true for some  $k \geq 1$ . Assume  $\sum_{i=1}^k (2i - 1) = k^2$ .
4. Induction (we need to prove this):

$$\begin{aligned}
 P(k+1) &= \sum_{i=1}^{k+1} (2i - 1) = (k+1)^2 \\
 &= \sum_{i=1}^k (2i - 1) + (2(k+1) - 1) \\
 &= k^2 + 2k + 1
 \end{aligned}$$

5. Proof:

$$\begin{aligned}
 \sum_{i=1}^{k+1} (2i - 1) &= \sum_{i=1}^k (2i - 1) + \sum_k^{k+1} (2i - 1) \\
 &= \sum_{i=1}^k (2i - 1) + (2(k+1) - 1) \\
 &= k^2 + (2k + 2 - 1) \\
 &= k^2 + 2k + 1
 \end{aligned}$$

## Example

Prove  $\forall n, n^3 - n \equiv 0 \pmod{3}$ . Let  $P(n)$  be the idea  $n^3 - n \equiv 0 \pmod{3}$ .

1. Basis:  $P(0)$  is the statement  $0^3 - 0 \equiv 0 \pmod{3}$ . We get  $0 \equiv 0 \pmod{3}$  from simplifying this statement, which is true.
2. Induction hypothesis: assume that  $P(k)$  is true for some  $k \geq 0$ . This means that  $k^3 - k \equiv 0 \pmod{3}$  is true.
3. Induction (we need to prove this):

$$(k+1)^3 - (k+1) \equiv 0 \pmod{3}$$

4. Proof:

$$\begin{aligned}
 (k+1)^3 - (k+1) &\equiv k^3 + 3k^2 + 3k + 1 - k - 1 \pmod{3} \\
 &\equiv (k^3 - k) + 3k^2 + 3k \\
 &\equiv 0 + 0k^2 + 0k \equiv 0
 \end{aligned}$$

If you have any questions, comments, or concerns, please contact me at  
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