# Sets

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## Countable and Uncountable Sets

Recall for a set $S$, $|S|$ is the cardinality of $S$, If $S$ has a finite number of objects, then $|S| \in N$. Consider the following six sets:

- $E$ is the set of positive even numbers less than or equal to 100.
- $\mathbb{Z}$ is the set of all integers.
- $U$ is the set of all real numbers between 0 and 1.
- $P$ is the set of all prime numbers less than or equal to 50.
- $\mathbb{Q}$ is the set of all rational numbers.
- $\mathbb{R}$ is the set of all real numbers.

Both $E$ and $P$ are finite sets with $|E| = 50$ and $|P| = 15$. The remaining four sets are all infinite. If $f : A \to B$ is 1-1, then $|A| \leq |B|$. Recall from functions that $f$ is 1-1 if every output is uniquely associated to 1 input. Also, for real-valued functions, $f$ is 1-1 if it passed the horizontal line test. If $A \subseteq B$, then $|A| \leq |B|$ by using $f(x) = x$ where $f : A \to B$.

### Schröder-Bernstein Theorem

If $f : A \to B$ is 1-1 and $g : B \to A$ is 1-1, then $|A| = |B|$. Using the above example, $|U| = |\mathbb{R}|$ since $f(x) = \frac{\arctan(x)+\frac{\pi}{2}}{\pi}$ where $f : \mathbb{R} \to U$ is 1-1 and $g(x) = x$ where $g : U \to \mathbb{R}$ is 1-1.

## Countably Infinite Sets

Set $A$ is **countably infinite** if $|A| = |\mathbb{Z}^+|$. Equivalently, there are 1-1 functions $f, g$ such that $f : A \to \mathbb{Z}^+$ and $g : \mathbb{Z}^+ \to A$. There is a good enumeration of $A = \{a_1, a_2, \ldots$ such that every number in $A$ shows up once and every number appears with a finite index. A set is **countable** if it is a finite set or countably infinite.
**Countably Infinite Sets**: $\mathbb{Z}^+, \mathbb{Z}, \mathbb{Q}, \mathbb{N} \times \mathbb{N}$
**Uncountable Sets**: $\mathbb{R}, (0, 1), P(\mathbb{Z}) : subsets\ of\ \mathbb{Z}$
If $A, B$ are both countable, then $A \cup B$ is countable.

## Enumerating a Countable Set

A good enumeration for a set $A$ involves listing out the values in $A$ with a predictable pattern so that every value in $A$ is listed at a finite step. For example, $\mathbb{Z} = \{0, 1, 2, 3, \ldots, -1, -2, -3, \ldots\}$ is not a good enumeration as we would have to go through all infinitely many positive integers before seeing any negative integers. A good enumeration for $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$ so that both positive integers and negative integers are reached in a finite number of steps.

## Subsets of Sets

Suppose $A \subseteq B$:

- If $B$ is countable, then $A$ is countable as well.
- If $B$ is uncountable, then $A$ is uncountable as well.

## Uncountable Sets

Two sets $A$ and $B$ have the same **cardinality**, written $A \sim B$, if there is a bijection between them. Let $A$ be a set:

- $A$ is **countable** if it is finite: $|A| \in \mathbb{Z}$
- $A$ is **countably infinite** if $A \sim \mathbb{Z}^+$
- $A$ is **uncountable** if it is not countable.

If you have any questions, comments, or concerns, please contact me at
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