

Sequences and Summations

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Sequences and Summations

Let a_n be the number of binary strings of length n with substring "10".

n	a_n
0	0
1	0
2	1
3	4
n	?

Recursive formula for a_n :

$$a_n = 2a_{n-1} + n - 1$$

Sequences

A **sequence** (called a_n or $f(n)$) is a function from \mathbb{N} to \mathbb{R} . Example:

1. $\{a_n\} = \{2, 5, 8, 11, 14, 19, \dots\}$
2. $\{b_n\} = \{2, 4, 8, 16, 32, 64, \dots\}$
3. $\{c_n\} = \{1, 2, 5, 10, 17, 26, \dots\}$

A **linear sequence** or **arithmetic sequence** has a constant difference d between adjacent terms.

$$\{a_0, a_0 + d, a_0 + 2d, a_0 + 3d, \dots\}$$

$$a_n = dn + a_0$$

Difference sequences:

$$\{a'_n\} = \{a_n - a_{n-1} \mid n \geq 1\}$$

$$\{a_n\} = \{2, 5, 8, 11, \dots\}$$

$$\{a'_n\} = \{5 - 2, 8 - 5, 11 - 8, \dots\} = \{3, 3, 3, \dots\}$$

$$\{c_n\} = \{1, 2, 5, 10, 17, 26, \dots\}$$

$$\{c'_n\} = \{1, 3, 5, 7, 9, \dots\}$$

$$\{c''_n\} = \{2, 2, 2, 2, 2, \dots\}$$

A **geometric sequence** has a constant ratio r , equal to $\frac{a_{n+1}}{a_n}$.

$$\{a_0, a_0r, a_0r^2, a_0r^3\}$$

$$a_n = a_0r^n$$

$$\{b_n\} = \{2, 4, 8, 16, \dots\}$$

$$= 2^n$$

$$\{b'_n\} = \{2, 4, 8, 16, \dots\}$$

$$\{b''_n\} = \{2, 4, 8, 16, \dots\}$$

Example

$$\{5, 15, 45, 135, \dots\} = \{a_n\}$$

$$r = \frac{20}{5} = \frac{80}{4} = 4$$

$$a_n = 5 \times 3^n$$

Example

Find a non-recursive formula for $a_n = 2a_{n-1} + n - 1$.

$$\begin{aligned}a_n &= A2^n + Bn + C \\a_{n-1} &= A2^{n-1} + B(n-1) + C \\a_n &= 2(a_{n-1}) + n - 1 \\A2^n + Bn + C &= 2(A2^{n-1} + B(n-1) + C) + n - 1 \\B &= -1 = C \\A &= 1 \\a_n &= 2^n - n - 1\end{aligned}$$

Summation Notation

$$\sum_{i=M}^N f(i) = f(M) + f(M+1) + \cdots + f(N)$$

$$\sum_{i=M}^N f(i) = \sum_{j=M}^N f(j) = \sum_{k=M}^N f(k)$$

Example

$$\sum_{i=1}^5 i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

$$\sum_{j=5}^8 \left(\frac{1}{j}\right) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$\sum_{k=0}^5 (2^k) = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 2^6 - 1$$

Summation Properties

1.

$$\sum_{i=M}^N (kf(i)) = k\left(\sum_{i=M}^N f(i)\right)$$

2.

$$\sum_{i=M}^N (f(i) + g(i)) = \sum_{i=M}^N f(i) + \sum_{i=M}^N g(i)$$

3.

$$\sum_{i=M}^N f(i) = \sum_{i=1}^N f(i) - \sum_{i=1}^{M-1} f(i)$$

Summation Formulas

1.

$$\sum_{i=1}^n 1 = n$$

2. Gauss's Formula

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

3.

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

4.

$$\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2} \right)^2$$

Geometric Sequence

$$\{a, ar, ar^2, ar^3, \dots\}$$

$$\sum_{i=0}^n (ar^i) = a \left(\frac{1 - r^{n+1}}{1 - r} \right)$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanagerd.tech