

Sets

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Functions

Let A and B be sets. A **function** f is a rule that assigns to each element $x \in A$ exactly one element $y \in B$, written $y = f(x)$. A is called the **domain** while B is called the **codomain**. The **range** of f , denoted $\text{ran}(f)$:

$$\text{ran}(f) = \{f(x) \in B \mid x \in A\}$$

$$\text{ran}(f) \subseteq B$$

Example

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = x^2$.

- Domain: \mathbb{R}
- Codomain: \mathbb{R}
- Range: $[0, \infty)$

Example

Let $g : \mathbb{R} \rightarrow [0, \infty)$ be defined as $g(x) = x^2$.

- Domain: \mathbb{R}
- Codomain: $[0, \infty)$
- Range: $[0, \infty)$

Function Equality

Two functions, $f : A \rightarrow B$ and $g : C \rightarrow D$ are **equal**, denoted $f = g$, if:

- $A = C$
- $B = D$
- $f(x) = g(x) \quad \forall x \in A$

Absolute Value

The **absolute value** of $x \in \mathbb{R}$, written $abs(x)$ or $|x|$, is a piecewise function defined as:

$$abs(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$
$$abs : \mathbb{R} \rightarrow \mathbb{R}$$

- Domain: \mathbb{R}
- Codomain: \mathbb{R}
- Range: $[0, \infty)$

Example

Let X be a set, $P(X)$ be the power set of X , and $S \subseteq X$. Let $f : P(X) \rightarrow P(X)$ be defined as $f(A) = A \cup S$.

- Domain: $P(X)$
- Codomain: $P(X)$
- Range: $\{B \subseteq X \mid B \supseteq S\}$

Function Images and Preimages

Let $f : A \rightarrow B$ be a function. If $S \subseteq A$, then the **image** of S , denoted $f(S)$, is defined as:

$$f(S) = \{f(x) : x \in S\}$$

If $T \subseteq B$, then the **preimage** of S , denoted $f^{-1}(S)$, is defined as:

$$f^{-1}(T) = \{x \in A \mid f(x) \in T\}$$

Example

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^2$. Determine:

1. $f([-1, 2]) = [0, 4]$
2. $f^{-1}(\{2\}) = \{\sqrt{2}, -\sqrt{2}\}$
3. $f^{-1}([-1, 2])$

$$\begin{aligned}f^{-1}(\{-1\}) &= \emptyset \\f^{-1}([-1, 0]) &= \emptyset \\f^{-1}(\{a, b\}) &= f^{-1}(\{a\}) \cup f^{-1}(\{b\}) \\f^{-1}([-1, 2]) &= f^{-1}([-1, 0]) \cup f^{-1}([0, 2]) \\&= \emptyset \cup [-\sqrt{2}, \sqrt{2}] \\&= [-\sqrt{2}, \sqrt{2}]\end{aligned}$$

One-To-One and Onto

Let $f : A \rightarrow B$, f is **one-to-one** if distinct elements in A have distinct images in B . To show a function is one-to-one:

$$\forall x, y \in A (x \neq y \rightarrow f(x) \neq f(y))$$

Contrapositive:

$$\forall x, y \in A (f(x) = f(y) \rightarrow x = y)$$

Onto

Let $f : A \rightarrow B$, f is **onto** if the range and codomain are the same.

$$B = \text{ran}(f) = f(A)$$

Example

Let $x \in \mathbb{R}$. The greatest integer of x , denoted $\lfloor x \rfloor$, is the largest integer less than or equal to x . $\lfloor x \rfloor = \text{floor}(x) = \text{floor} : \mathbb{R} \rightarrow \mathbb{R}$.

$$\begin{aligned}\lfloor 2.1 \rfloor &= 2 \\ \lfloor \pi \rfloor &= 3 \\ \lfloor -2.1 \rfloor &= -3\end{aligned}$$

Is $\lfloor x \rfloor$ one-to-one?

$$\begin{aligned}\lfloor 1.1 \rfloor &= \lfloor 1 \rfloor \\ 1.1 &\neq 1\end{aligned}$$

Therefore, it is not one-to-one.

Is $\lfloor x \rfloor$ onto?

No, there is no $x \in \mathbb{R}$ such that $\lfloor x \rfloor = 2.5$.

Example

Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $f(n) = n^3$. Is f one-to-one?

- Suppose $x, y \in \mathbb{Z}$ and $f(x) = f(y)$.
- This implies $x^3 = y^3$.
- Taking the cube root yields $x = y$.

Is f onto?

No, since $2 \in \mathbb{Z}$ but there is no $x \in \mathbb{Z}$ such that $x^3 = 2$.

Example

Let X be a set and $X \subseteq X$. Define $f : P(X) \rightarrow P(X)$ as $f(A) = A \cup S$. Is f one-to-one?

No, since $f(\emptyset) = f(S)$ but $\emptyset \neq S$.

$$\begin{aligned}f(\emptyset) &= \emptyset \cup S = S \\ f(S) &= S \cup S = S\end{aligned}$$

Example

Let $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $f(m, n) = m + n$.

Is f one-to-one?

No, $f(2, 3) = f(3, 2)$ but $(2, 3) \neq (3, 2)$

Is f onto?

Yes, take any $n \in \mathbb{Z}$. Find an ordered pair in $\mathbb{Z} \times \mathbb{Z}$ that maps to it. Since $f(n - 1, 1) = (n - 1) + 1 = n$ and $(n - 1, 1) \in \mathbb{Z} \times \mathbb{Z}$, then f is onto.

Identity Mapping

Let A be a set. The function $1_A : A \rightarrow A$ is defined as:

$$1_A(x) = x$$

This is called the **identity mapping**.

Bijections

A function is a **bijection** if it is one-to-one and onto. A “bijection” is a “one-to-one correspondence”.

Composition

Let $f : A \rightarrow A$ and $g : B \rightarrow C$. The **composition** of f and g , denoted $g \circ f$, is $g \circ f : A \rightarrow C$ and defined as:

$$(g \circ f)(x) = g(f(x))$$

Inverse

Let f be a bijection. ($f : A \rightarrow B$). The **inverse** of f is the unique function $g : B \rightarrow A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. Traditionally, we represent g as f^{-1} . Note that $f^{-1} \neq \frac{1}{f}$.

Example

Let $A = \{1, 2, 3\}$, $B = \{x, y, z\}$, $C = \{\alpha, \beta, \gamma\}$. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be defined as:

$$\begin{aligned} f &= \{(1, x), (2, y), (3, x)\} \\ g &= \{(x, \gamma), (y, \alpha), (z, \beta)\} \end{aligned}$$

Find $f \circ g$:

$$\begin{aligned} f \circ g &= \text{undefined} \\ g \circ f &= \{(1, \gamma), (2, \alpha), (3, \beta)\} \end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech