Methods of Proofs

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Discrete Math for Computing: January 2017 - May 2017

Methods of Proofs

Exhaustive Proofs and Proof by Cases

Sometimes, to prove $p \rightarrow q$ is true, it is convenient to use $P_1 \vee P_2 \vee P_3 \vee \cdots \vee P_n$ instead of p as a hypothesis. These types of proofs are called **proofs by exhaustion**.

A **proof by cases** is one which covers all possible cases that arise in a theorem.

An Exhaustive Proof

Prove that $(n+1)^3 \geq 3^n$ if n is a positive integer $n \leq 4$. Proof:

- $n = 1 : 2^3 = 8 \geq 3^1$
- $n = 2 : 3^3 = 27 \geq 3^2$
- $n = 3 : 4^3 = 64 \geq 3^3$
- $n = 4 : 5^3 = 125 \geq 3^4$

So, by exhaustion of all cases, this proposition is true for $a = 1, 2, 3, 4$.

A Proof By Cases

If n is an integer, $n^2 \geq n$. Proof:

- Case 1: $n = 0$

$$0^2 = 0 \geq 0$$

- Case 2: $n > 0$

$$\begin{aligned} n &\geq 1 \\ (n)(n) &\geq (1)(n) \\ n^2 &\geq n \end{aligned}$$

- Case 3: $n \leq -1$

$$n^2 \geq 0$$

$$\therefore n^2 \geq n$$

Since all integers, fall into 1 of these cases, we conclude our proof.

Example

Recall:

$$|a| = \begin{cases} a \ if \ a \geq 0 \\ -a \ if \ a < 0 \end{cases}$$

Prove by cases that: $|xy| = |x||y|$. There are four cases to consider.

- Case 1: $x \geq 0, y \geq 0$

$$\begin{aligned} xy &\geq 0 \\ |xy| &= xy \\ &= |x||y| \end{aligned}$$

- Case 2: $x > 0, y < 0$

$$\begin{aligned} xy &< 0 \\ |xy| &= -(xy) \\ &= x(-y) \\ &= |x||y| \end{aligned}$$

- Case 3: $x < 0, y > 0$
  Without loss of generality, this is the same as case 2.

- Case 4: $x < 0, y < 0$

$$\begin{aligned} xy &\geq 0 \\ |xy| &= xy \\ &= (-x)(-y) \\ &= |x||y| \end{aligned}$$

In general, using without loss of generality states that the proof of a current case is identical to a previous case (perhaps with variables switched).

Example

Show that if x and y are integers and both xy and $x + y$ are even, then both x and y are even. Proof:

- Suppose x and y are both not even. x is odd or y is odd (or both).
- Without loss of generality, assume that x is odd. Thus $x = 2k + 1; k \in \mathbb{Z}$.
- To show xy is odd or $x + y$ is odd, consider two cases:
  1. y is even. Then $y = 2n; n \in \mathbb{Z}$.

     $$x + y = 2k + 1 + 2n = 2(k + n) + 1$$

     $x + y$ is odd.
  2. y is odd. Then $y = 2m + 1; m \in \mathbb{Z}$.

     $$xy = (2k+1)(2m+1) = 2km + 2k + 2m + 1 = 2(km + k + m) + 1$$

     xy is odd.

Uniqueness Proofs

To show uniqueness, we show if $x \neq y$ then y does not have some desired property.

Example

Show if a, b are real numbers, with $a \neq 0$, then there exists a unique real number r such that $ar + b = 0$.

Proof of Existence

Let $r = \frac{-b}{a}$. Then $ar + b = a(\frac{-b}{a}) + b = 0$. So such an r exists.

Proof of Uniqueness

Suppose there is some s such that $as + b = 0$. $ar + b = as + b$ where $r = \frac{b}{-a}$. Therefore, $ar = as$ and $r = s$. If $r \neq s$ then $as + b \neq 0$. So $r = \frac{-b}{a}$ is the only such solution.

Example

Show if $x + y \geq 2$ (where x and y are real numbers) then $x \geq 1$ or $y \geq 1$. Proof by contraposition:

- Suppose $\neg(x \geq 1 \textit{ or } y \geq 1)$.
- Then $x < 1$ and $y < 1$.
- We must show that $x + y < 2$, but $x < 1$ and $y < 1$.
- Then $x + y < 1$.

Example

Show that $n^2 + 1 \geq 2^n$ for any positive integer $n \leq 4$. Proof by exhaustion: Verify that $n^2 + 1 \geq 2^n$ for $n = 1, 2, 3, 4$.

1. $n = 1 : 1^2 + 1 \geq 2^1$
2. $n = 2 : 2^2 + 1 \geq 2^2$
3. $n = 3 : 3^2 + 1 \geq 2^3$
4. $n = 4 : 4^2 + 1 \geq 2^4$

Example

Show that every odd integer is a difference of two squares. Proof:

- Let $n = 2k + 1; k \in \mathbb{Z}$.

$$n = 2k + 1 + k^2 - k^2$$
$$n = k^2 + 2k + 1 - k^2$$
$$n = (k + 1)^2 - k^2$$

- Thus, we've expressed n as a difference of two squares.

If you have any questions, comments, or concerns, please contact me at
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