Nested Quantifiers

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Nested Quantifiers

In section 1.4, we avoided nested quantifiers where one was in the scope of the other.

Example

Assume that the domain of x and y are real numbers:

1. $$\forall x \forall y(x + y = y + x)$$

   For all real numbers x and y, $x + y = y + x$

2. $$\forall x \exists y(x + y = 0)$$

   For all real numbers x, there exists a y such that $x + y = 0$.

Translate the following into English:

$$\forall x \forall y((x > 0) \wedge (y < 0)) \rightarrow (xy < 0)$$

For all real x and all real y if $x > 0$ and $y > 0$, then $xy > 0$. The product of a positive real number and negative real number is negative.

Order of Quantifiers

The order of quantifiers is essential, unless the quantifiers have all the same "type" (Either the quantifiers are all universal or all existential).

Example

$$P(x, y) : x + y = y + x$$

What are the truth values when $domain = \mathbb{R}$?

1. $\forall x \forall x\ P(x, y)$ **True**
2. $\forall y \forall x\ P(x, y)$ **True**

$$Q(x, y) : x + y = 0$$

Domain is $\mathbb{R}$, what are the truth values?

1. $\exists y \forall x\ Q(x, y)$ There is a real number y such that for all real x, $x+y = 0$. **False**
2. $\forall x \exists y\ Q(x, y)$ For all real x, there is a real number y such that $x+y = 0$. **True**

Example

Translate the following into a logical expression:

1. The sum of two positive integers is always positive.

$$\forall x \forall y ((x > 0) \land (y > 0)) \rightarrow (xy > 0)$$

2. Every real number except 0 has a multiplicative inverse.

$$\forall x ((x \neq 0) \rightarrow \exists y (xy = 1))$$

Example

Write $\neg(\forall x \exists y (xy = 1))$ without any negations in the expression.

$$\exists x \neg(\exists y (xy = 1))$$

$$\exists x \forall y \neg(xy = 1)$$

$$\exists x \forall y (xy \neq 1)$$