Predicates and Quantifiers

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Predicates and Quantifiers

Propositional logic cannot express the meaning of many mathematical or English statements.

Predicate Logic

Statements with variables are called **predicates**. For example, $x > 9, x + 2y = 1, x^2 + y^2 = z^2$. We can view predicates as propositional functions from some domain to some range. A predicate in the variable x will be denoted as $P(x)$. The domain of a predicate is often called the **universe of discourse**.

$$P(x) : x > 9$$

$$P(2) : 2 > 9 \ is \ false.$$
$$P(4) : 4 > 9 \ is \ false.$$
$$P(10) : 10 > 9 \ is \ true.$$

$$Q(x, y) : x = y + 2$$

$$Q(1, 2) : 1 = 2 + 2 \ is \ false.$$
$$Q(3, 0) : 3 = 0 + 2 \ is \ false.$$
$$Q(2, 0) : 2 = 0 + 2 \ is \ true.$$

Quantifiers

We will use the following three quantifiers in this class:

1. **Universal Quantifier**: The universal quantifier of $P(x)$ is the statement "P(x) for all values x in the domain". Notated as $\forall x P(x)$.

2. **Existential Quantifier**: The existential quantifier of a statement $P(x)$ is "There exists an element x in the domain so that P(x) holds". Notated as $\exists x P(x)$.

3. **Uniqueness Quantifier**: The uniqueness quantifier of $P(x)$ is the statement "there exists a unique x such that P(x) holds". Notated as $\exists! x$.

Example Problem

What are the truth values of the following if the universe of discourse is all positive integers not exceeding four?

1. $\exists x P(x) \; where \; P(x) : x^2 > 10$

   $$Domain = \{1, 2, 3, 4\}$$

   $$1^2 \not> 10$$

   $$2^2 \not> 10$$

   $$3^2 \not> 10$$

   $$4^2 > 10$$

   Therefore, this statement is true.

2. $\forall x P(x) \; where \; P(x) : x^2 > 10$

   $$Domain = \{1, 2, 3, 4\}$$

   $$1^2 \not> 10$$

   Therefore, this statement is false.

Precedence of Quantifiers

The quantifiers $\forall x$ and $\exists x$ have higher precedence that those from propositional logic. When $\forall x$ and $\exists x$ are in statements together, things get tricky.

$$\forall x P(x) \vee Q(x) \equiv (\forall x P(x)) \vee Q(x)$$

Binding Variables

When a quantifier is used on the variable x, we say this instance of x is a **bound variable**. A variable that is not bound is called a **free variable**. The part of a logical expression in which a quantifier is applied is called the quantifier's **scope**.

$$\exists x(x + y = 1)$$

x is a bound variable while y is a free variable.

Logical Equivalences with Quantifiers

Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter what predicates are used and what universe of discourse is used. **You must prove things in full generality**.

Negation of Quantifiers

$$\neg \forall x P(x) \equiv \forall x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

We will use these facts often, especially for De Morgan's laws for predicate logic and set theory.

De Morgan's Laws for Quantifiers

Negation	Equivalent	When it is true, then...	When it is false, then...
$\neg \exists x P(x)$	$\forall x \neg P(x)$	P(x) is false for all x	There is an x when P(x) is true
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an x when P(x) is false	P(x) is true for all x

Example

Rewrite in simple form:

$$\forall x(x^2 > x) \Rightarrow \neg(\forall x(x^2 > x))$$

$$\exists x \neg(x^2 > x) \equiv \exists x(x^2 \leq x)$$

Rewrite in simple form:

$$\exists x(x^2 = 2) \Rightarrow \neg(\exists x(x^2 = 2))$$

$$\forall x \neg(x^2 = 2) \equiv \forall x(x^2 \neq 2)$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech