

Taylor and Maclaurin Series

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Taylor and Maclaurin Series

Here is a function $f(x)$ that can be represented as a power series:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n(x-a)^n \\ &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots \end{aligned}$$

If we assume that the derivatives of $f(x)$ in every order exist, then we can solve for the coefficients c_n . For example:

$$f(a) = c_0$$

Using the first derivative:

$$\begin{aligned} f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 \\ f'(a) &= c_1 \end{aligned}$$

Using the second derivative:

$$\begin{aligned} f''(x) &= 2c_2 + (3)(2)c_3(x-a) + (4)(3)(x-a)^2 + (5)(4)(x-a)^3 \\ f''(a) &= 2c_2 \\ c_2 &= \frac{f''(a)}{2} \end{aligned}$$

Using the third derivative:

$$\begin{aligned} f'''(x) &= (3)(2)(1)c_3 + (4)(3)(2)c_4(x-a) + (5)(4)(3)c_5(x-a)^2 \\ f'''(a) &= (3)(2)c_3 \\ c_3 &= \frac{f'''(a)}{(3)(2)} \end{aligned}$$

It follows that a general form of the coefficient c_n is:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Provided a power series representation for the function $f(x)$ about $x = a$ exists, the Taylor Series for $f(x)$ about $x = a$ is:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \dots \end{aligned}$$

If we use $a = 0$, then the Taylor Series is known as the Maclaurin Series for $f(x)$:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots \end{aligned}$$

Example 1

Find the Taylor Series for $f(x) = e^x$ about $x = 0$:

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f^{(n)}(x) = e^x$$

$$f^{(n)}(0) = e^0 = 1$$

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \end{aligned}$$

We can actually derive the value of e by taking $f(1)$:

$$f(1) = e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots$$

Example 2

$$\begin{aligned}\sin(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\ &= 0 + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ \cos(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ e^{ix} &= \cos(x) + i \sin(x)\end{aligned}$$

Example 3

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{1 + x - e^x} \\ \lim_{x \rightarrow 0} \frac{1 - \cos(x)}{1 + x - e^x} &= \lim_{x \rightarrow 0} \frac{1 - [1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots]}{1 + x - [1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots]} \\ &= \lim_{x \rightarrow 0} \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \dots}{-\frac{x^2}{2!} - \frac{x^3}{3!}} \\ &= \lim_{x \rightarrow 0} \frac{\frac{x^2}{2}}{-\frac{x^2}{2}} \left[\frac{1 - \frac{x^2}{2} - \dots}{1 + \frac{x}{3}} \right] \\ &= -1\end{aligned}$$

Example 4

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\sin(x) - x - x^{\frac{3}{6}}}{x^3} \\ \lim_{x \rightarrow \infty} \frac{\sin(x) - x - x^{\frac{3}{6}}}{x^3} &= \lim_{x \rightarrow 0} \frac{(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots) - x - x^{\frac{3}{6}}}{x^3} \\ &= \lim_{x \rightarrow 0} \frac{-1}{3} + \frac{x^2}{5!} - \dots \\ &= -\frac{1}{3}\end{aligned}$$

Example 5

$$y = e^x \ln(1 - x)$$

$$\begin{aligned} y &= e^x \ln(1 - x) \\ &= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots\right) \left(-x - \frac{x^2}{2} + \dots\right) \end{aligned}$$

Example 6

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1} (2n+1)!}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1} (2n+1)!} &= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} \\ &= \frac{\pi}{4} - \frac{\left(\frac{\pi}{4}\right)^3}{3!} + \dots \\ &= \sin\left(\frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2} \end{aligned}$$

Example 7

$$\begin{aligned} 1 - \ln(2) + \frac{(\ln(2))^2}{2!} - \frac{(\ln(2))^3}{3!} + \dots \\ = e^{-\ln(2)} = \frac{1}{2} \end{aligned}$$

Binomial Series

Binomial Theorem: If n is any positive integer, then:

$$\begin{aligned}(a + b)^n &= \sum_{i=0}^{\infty} \binom{n}{i} a^{n-i} b^i \\ &= a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \dots + nab^{n-1} + b^n\end{aligned}$$

where :

$$\begin{aligned}\binom{n}{i} &= \frac{n(n-1)(n-2)\dots(n-i+1)}{i!}; \quad i = 1, 2, 3, \dots, n \\ \binom{n}{0} &= 1\end{aligned}$$

Example 1

$$(1 + x)^k \quad |x| < 1$$

$$\begin{aligned}(1 + x)^k &= \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \\ &= 1 + \frac{kx}{1!} + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)\dots(k-(n-1))x^n}{n!}\end{aligned}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

If any errors are found, please contact me at alvin.lin.dev@gmail.com