

Power Series

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Power Series

A power series is any series that can be written in the form:

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

where a is any number and c is a coefficient of the series. There exists a number R such that the power series converges for $|x-a| < R$ and diverges for $|x-a| > R$. This is called the radius of convergence. We are always guaranteed convergence for $x = a$ because:

$$\sum_{n=0}^{\infty} c_n(a-a)^n = \sum_{n=0}^{\infty} c_n(0)^n = c_0(0)^0 + \sum_{n=1}^{\infty} c_n(0)^n = c_0 + \sum_{n=1}^{\infty} 0 = c_0$$

Let $a = 0$:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

This can converge for a single value of x , for multiple values of x , or for all x .

Example

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$$

$$\begin{aligned}
a_n &= \frac{x^n}{n} \\
a_{n+1} &= \frac{x^{n+1}}{\sqrt{n+1}} \\
\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{\sqrt{n+1}}}{\frac{x^n}{n}} \right| \\
&= \lim_{n \rightarrow \infty} |x| \frac{\sqrt{n+1}}{\sqrt{n}} \\
&= \lim_{n \rightarrow \infty} |x| \sqrt{1 + \frac{1}{n}} \\
&= |x|
\end{aligned}$$

$\sum \frac{x^n}{\sqrt{n}}$ converges when $|x| < 1$. Thus $(-1, 1)$ is the radius of convergence.

Example 2

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{x^n}{n!} \\
a_n &= \frac{x^n}{n!} \\
a_{n+1} &= \frac{x^{n+1}}{(n+1)!} \\
\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| \\
&= \lim_{n \rightarrow \infty} |x| \left(\frac{n!}{(n+1)!} \right) \\
&= \lim_{n \rightarrow \infty} |x| \frac{1}{n+1} \\
&= 0
\end{aligned}$$

This series converges for all $x \in \mathbb{R}$. Thus the radius of convergence is ∞ .

Practice Problem 4

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$$

$$\begin{aligned}
a_n &= (-1)^n \frac{x^n}{n+1} \\
a_{n+1} &= (-1)^{n+1} \frac{x^{n+1}}{n+2} \\
\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(-1)^n \frac{x^n}{n+1}}{(-1)^{n+1} \frac{x^{n+1}}{n+2}} \right| \\
&= \lim_{n \rightarrow \infty} |x| \left(\frac{n+1}{n+2} \right) \\
&= \lim_{n \rightarrow \infty} |x| \left(\frac{n}{n} \right) \left(\frac{1 + \frac{1}{n}}{1 + \frac{2}{n}} \right) \\
&= |x|
\end{aligned}$$

$\sum \frac{x^n}{n+1}$ converges when $|x| < 1$. Thus $(-1, 1)$ is the radius of convergence.

If any errors are found, please contact me at alvin.lin.dev@gmail.com