

Absolute Convergence and the Ratio and Root Tests

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Absolute Convergence and the Ratio and Root Tests

Definition: $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges. If $\sum |a_n|$ diverges but $\sum a_n$ converges, then $\sum a_n$ is conditionally convergent.

The Ratio Test

Let $\sum a_n$ be given series. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

1. If $L < 1$, then $\sum a_n$ converges absolutely.
2. If $L > 1$, then $\sum a_n$ diverges.
3. If $L = 1$, then the test fails.

Example 1

$$\sum \frac{n^2}{2^n}$$

$$\begin{aligned}
a_n &= \frac{n^2}{2^n} \\
a_{n+1} &= \frac{(n+1)^2}{2^{n+1}} \\
\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} \\
&= \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{2n^2} \\
&= \lim_{n \rightarrow \infty} \left(\frac{n^2}{n^2} \right) \left(\frac{1 + \frac{2}{n} + \frac{1}{n^2}}{2} \right) \\
&= \frac{1}{2}
\end{aligned}$$

Since $\frac{1}{2} < 1$, $\sum a_n$ is absolutely convergent.

Example 2

$$\begin{aligned}
&\sum \frac{(-10)^n}{n!} \\
a_n &= \frac{(-10)^n}{n!} \\
a_{n+1} &= \frac{(-10)^{n+1}}{(n+1)!} \\
\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{\frac{(-10)^{n+1}}{(n+1)!}}{\frac{(-10)^n}{n!}} \\
&= \lim_{n \rightarrow \infty} \frac{-10}{n+1} \\
&= 0
\end{aligned}$$

Since $0 < 1$, $\sum a_n$ is absolutely convergent.

Example 3

$$\sum a_n = 1 - \frac{1 \times 3}{3!} + \frac{1 \times 3 \times 5}{5!} - \frac{1 \times 3 \times 5 \times 7}{7!} + \dots$$

$$\begin{aligned}
a_n &= (-1)^{n-1} \frac{1 \times 3 \times 5 \times \dots (2n-1)}{(2n-1)!} \\
a_{n+1} &= (-1)^n \frac{1 \times 3 \times 5 \times \dots (2n-1) \times (2n+1)}{(2n+1)!} \\
\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{1 \times 3 \times 5 \times \dots (2n-1) \times (2n+1)}{(2n+1)!} \times \frac{(2n-1)!}{1 \times 3 \times 5 \times \dots (2n-1)} \\
&= \lim_{n \rightarrow \infty} \frac{2n+1}{(2n)(2n+1)} \\
&= \lim_{n \rightarrow \infty} \frac{1}{2n} \\
&= 0
\end{aligned}$$

Since $0 < 1$, $\sum a_n$ is absolutely convergent.

Example 4

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n} \\
a_n &= \frac{n^2 2^{n-1}}{(-5)^n} \\
a_{n+1} &= \frac{(n+1)^2 2^n}{(-5)^{n+1}} \\
\left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{\frac{(n+1)^2 2^n}{(-5)^{n+1}}}{\frac{n^2 2^{n-1}}{(-5)^n}} \right| \\
&= \frac{(n+1)^2 2^n 5^n}{5^{n+1} n^2 2^{n-1}} \\
&= \frac{2}{5} \left(\frac{n+1}{n} \right)^2 \\
&= \frac{2}{5} \left(1 + \frac{1}{n} \right)^2 \\
&= \frac{2}{5}
\end{aligned}$$

Since $\frac{2}{5} < 1$, $\sum a_n$ is absolutely convergent.

The Root Test

Given $\sum a_n$:

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = L$$

1. If $L < 1$, the series is absolutely convergent.
2. If $L > 1$, the series diverges.
3. If $L = 1$, the test fails.

Example 1

$$\sum_{n=1}^{\infty} \left(\frac{n^2 + 1}{2n^2 + 1}\right)^n$$

$$\begin{aligned} a_n &= \left(\frac{n^2 + 1}{2n^2 + 1}\right)^n \\ \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n^2 + 1}{2n^2 + 1}\right)^n} &= \lim_{n \rightarrow \infty} \frac{n^2 + 1}{2n^2 + 1} \\ &= \lim_{n \rightarrow \infty} \left(\frac{n^2}{n^2}\right) \left(\frac{1 + \frac{1}{n^2}}{2 + \frac{1}{n^2}}\right) \\ &= \frac{1}{2} \end{aligned}$$

Since $\frac{1}{2} < 1$, $\sum a_n$ is absolutely convergent.

Example 2

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$$

$$\begin{aligned} a_n &= \left(1 + \frac{1}{n}\right)^{n^2} \\ \lim_{n \rightarrow \infty} \sqrt[n]{\left(1 + \frac{1}{n}\right)^{n^2}} &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \\ &= e \end{aligned}$$

Since $e > 1$, $\sum a_n$ diverges.

Example 3

$$\sum_{n=1}^{\infty} \left(\frac{-2n}{n+2}\right)^{7n}$$

$$\begin{aligned} a_n &= \left(\frac{-2n}{n+2}\right)^{7n} \\ \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{-2n}{n+2}\right)^{7n}} &= \lim_{n \rightarrow \infty} \left(\frac{2n}{n+2}\right)^7 \\ &= \lim_{n \rightarrow \infty} \left(\frac{n}{n}\right)^7 \left(\frac{2}{1+\frac{2}{n}}\right)^7 \\ &= 2^7 \end{aligned}$$

Since $2^7 > 1$, $\sum a_n$ diverges.

If any errors are found, please contact me at alvin.lin.dev@gmail.com