

The Comparison Tests

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The Comparison Tests

If $\sum a_n$ and $\sum b_n$ are two series with positive entries:

1. If $\sum b_n$ converges and $a_n \leq b_n$, then $\sum a_n$ converges
2. If $\sum a_n$ diverges and $a_n \leq b_n$, then $\sum b_n$ diverges
3. If $\sum b_n$ diverges and $b_n \leq a_n$, then $\sum a_n$ diverges

Example 1

$$\sum \frac{n}{2n^3 + 1}$$
$$\frac{n}{2n^3 + 1} \leq \frac{n}{2n^3 - 1} = \frac{n}{2n^3} = \frac{1}{2n^2}$$

Integral test:

$$\begin{aligned} \int_1^{\infty} \frac{1}{2x^2} dx &= \lim_{b \rightarrow \infty} \left[-\frac{x^{-1}}{2} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{2b} + \frac{1}{2} \right] \\ &= \frac{1}{2} \quad (\text{Convergent}) \end{aligned}$$

Therefore, since $\sum \frac{n}{2n^3+1} \leq \sum \frac{1}{2n^2}$ and $\sum \frac{1}{2n^2}$ is convergent, $\sum \frac{n}{2n^3+1}$ is also convergent.

Example 2

$$\sum \frac{n+1}{n\sqrt{n}}$$
$$\frac{n+1}{n\sqrt{n}} > \frac{n}{n\sqrt{n}}$$
$$\frac{1}{\sqrt{n}} = \infty \quad \therefore \sum \frac{n+1}{n\sqrt{n}} \text{ diverges}$$

Example 3

$$\sum \frac{9^n}{3+10^n}$$
$$\frac{9^n}{3+10^n} < \frac{9^n}{3+10^n-3} = \left(\frac{9}{10}\right)^n$$
$$\sum \left(\frac{9}{10}\right)^n = \frac{a}{1-r} = \frac{\frac{9}{10}}{1-\frac{9}{10}} = 9$$
$$\sum \left(\frac{9}{10}\right)^n \text{ converges} \quad \therefore \sum \frac{9^n}{3+10^n} \text{ diverges}$$

Practice Problem 4

$$\sum \frac{n^3}{n^4-1}$$
$$\frac{n^3}{n^4-1} > \frac{n^3}{n^4} = \frac{1}{n}$$
$$\frac{1}{n} = \infty \quad \therefore \sum \frac{n^3}{n^4-1} \text{ diverges}$$

Practice Problem 8

$$\sum \frac{4+3^n}{2^n}$$
$$\frac{4+3^n}{2^n} > \frac{3^n}{2^n}$$
$$\sum \frac{3^n}{2^n} = \infty \quad \therefore \sum \frac{4+3^n}{2^n} \text{ diverges}$$

Practice Problem 10

$$\sum \frac{n^2 - 1}{3n^4 + 1}$$
$$\frac{n^2 + 1}{3n^4 + 1} < \frac{n^2}{3n^4 + 1} < \frac{n^2}{3n^4} = \frac{1}{3n^2}$$
$$\sum \frac{1}{3n^2} < \infty \quad \therefore \quad \sum \frac{n^2 - 1}{3n^4 + 1} \text{ converges}$$

Practice Problem 12

$$\sum \frac{1 + \sin(n)}{10^n}$$
$$\frac{1 + \sin(n)}{10^n} \leq \frac{2}{10^n}$$
$$\sum \frac{2}{10^n} = \frac{a}{1 - r} < \infty \quad \therefore \quad \sum \frac{1 + \sin(n)}{10^n} \text{ converges}$$

Practice Problem 13

$$\sum \frac{\arctan(n)}{n^{1.2}}$$
$$\frac{\arctan(n)}{n^{1.2}} < \frac{\frac{\pi}{2}}{n^{1.2}}$$
$$\sum \frac{\frac{\pi}{2}}{n^{1.2}} < \infty \quad \therefore \quad \sum \frac{\arctan(n)}{n^{1.2}} \text{ converges}$$

Practice Problem 14

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3 + 1}}{3n^3 + 4n^2 + 2}$$
$$a_n = \frac{\sqrt{n^3 + 1}}{3n^3 + 4n^2 + 2}$$
$$b_n = \frac{n^{\frac{3}{2}}}{3n^3} = \frac{1}{3n^{\frac{3}{2}}} \quad (\text{converges})$$

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \frac{\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}}{\frac{1}{3n^{\frac{3}{2}}}} \\
&= \lim_{n \rightarrow \infty} \frac{3n^3}{3n^3+4n^2+2} \frac{\sqrt{n^3+1}}{\sqrt{n^3}} \\
&= 1
\end{aligned}$$

Therefore, $\sum a_n$ converges (p-series test).

Practice Problem 15

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n-1}{2n+1} \\
a_n &= \frac{n-1}{2n+1} \\
\lim_{n \rightarrow \infty} a_n &= \frac{1}{2} \neq 0 \quad \therefore \sum_{n=1}^{\infty} \frac{n-1}{2n+1} \text{ diverges}
\end{aligned}$$

If any errors are found, please contact me at alvin.lin.dev@gmail.com