

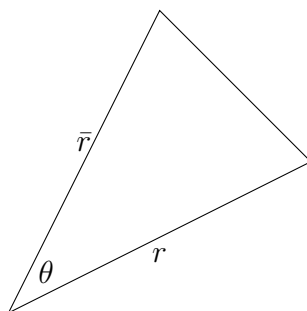
Areas and Lengths in Polar Coordinates

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Areas and Lengths in Polar Coordinates

Discrete Area



$$\begin{aligned}A_i &= \frac{1}{2}r^2\theta \\A &= \sum_{i=1}^n A_i \\&= \sum_{i=1}^n \frac{1}{2}r^2\Delta\theta \\&= \frac{1}{2} \int_a^b r^2 d\theta\end{aligned}$$

Arc Length

$$L = \int_a^b \sqrt{x^2 + y^2} d\theta$$

$$r = f(\theta) \quad \text{compared to} \quad y = f(x)$$

$$x = r \cos(\theta) = f(\theta) \cos(\theta) \implies \frac{dx}{d\theta} = -f(\theta) \sin(\theta) + f'(\theta) \cos(\theta)$$

$$y = r \sin(\theta) = f(\theta) \sin(\theta) \implies \frac{dy}{d\theta} = f'(\theta) \sin(\theta) + f(\theta) \cos(\theta)$$

$$\begin{aligned} x^2 + y^2 &= (-r \sin(\theta) + \frac{dr}{d\theta} \cos(\theta))^2 + (\frac{dr}{d\theta} \sin(\theta) + r \cos(\theta))^2 \\ &= r^2 \sin^2(\theta) + (\frac{dr}{d\theta})^2 \cos^2(\theta) - 2r \frac{dr}{d\theta} \sin(\theta) \cos(\theta) + \\ &\quad (\frac{dr}{d\theta})^2 \sin^2(\theta) + r^2 \cos^2(\theta) + 2r \frac{dr}{d\theta} \sin(\theta) \cos(\theta) \\ &= r^2(\cos^2(\theta) + \sin^2(\theta)) + (\frac{dr}{d\theta})^2(\cos^2(\theta) + \sin^2(\theta)) \\ &= r^2 + (\frac{dr}{d\theta})^2 \end{aligned}$$

$$L = \int_a^b \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$$

Example 1

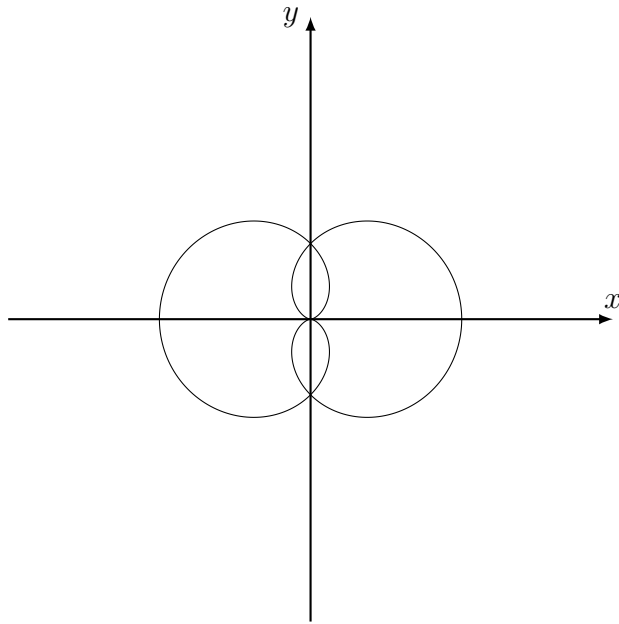
$$r = 2 - \sin(\theta)$$

$$\begin{aligned}
A &= \frac{1}{2} \int_0^{2\pi} r^2 d\theta \\
&= \frac{1}{2} \int_0^{2\pi} (2 - \sin(\theta))^2 d\theta \\
&= \frac{1}{2} \int_0^{2\pi} 4 + \sin^2(\theta) - 4 \sin(\theta) d\theta \\
&= \frac{1}{2} \int_0^{2\pi} 4 + \left(\frac{1 - \cos(2\theta)}{2}\right) - 4 \sin(\theta) d\theta \\
&= \frac{1}{2} \left[4\theta + \frac{\theta}{2} - \frac{\sin(2\theta)}{4} + 4 \cos(\theta) \right]_0^{2\pi} \\
&= \frac{1}{2} \left[\frac{9}{2} 2\pi - \frac{\sin 4\pi}{4} + 4 \cos(2\pi) - \left(-\frac{\sin(\theta)}{4} + 4 \cos(\theta)\right) \right] \\
&= \frac{1}{2} [9\pi + 4 - 4] \\
&= \frac{9\pi}{2}
\end{aligned}$$

Problem 30

Find the integral for the area in common between the two functions:

$$r = 1 + \cos(\theta) \quad r = 1 - \cos(\theta)$$

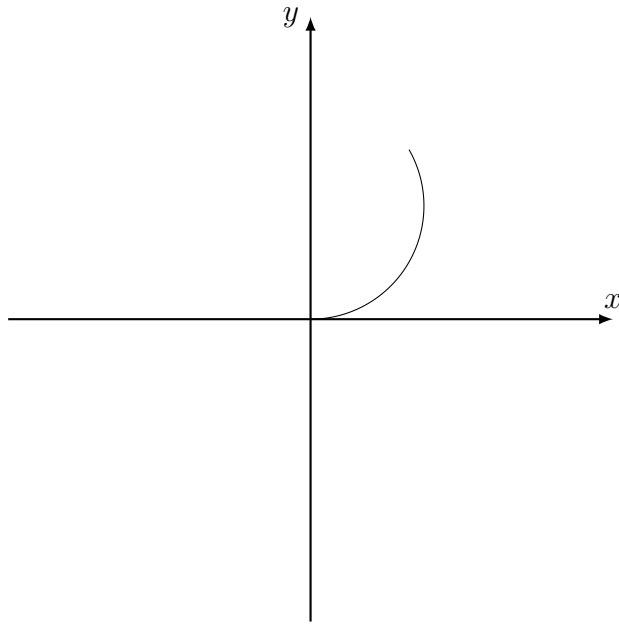


$$A = (4) \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos(\theta))^2 d\theta$$

Problem 35

Find the arc length:

$$r = 3 \sin(\theta) \quad \theta \in \left[0, \frac{\pi}{3}\right]$$



$$\begin{aligned}
 L &= \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\
 &= \int_0^{\frac{\pi}{3}} \sqrt{(3 \sin(\theta))^2 + (3 \cos(\theta))^2} d\theta \\
 &= \int_0^{\frac{\pi}{3}} \sqrt{9(\sin^2(\theta) + \cos^2(\theta))} d\theta \\
 &= \int_0^{\frac{\pi}{3}} 3 d\theta \\
 &= \left[3\theta \right]_0^{\frac{\pi}{3}} \\
 &= \pi
 \end{aligned}$$

Practice Problem 46

$$r = e^{2\theta} \quad \theta \in [0, 2\pi]$$

$$\begin{aligned}
L &= \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\
&= \int_0^{2\pi} \sqrt{(e^{2\theta})^2 + (2e^{2\theta})^2} d\theta \\
&= \int_0^{2\pi} \sqrt{e^{4\theta} + 4e^{4\theta}} d\theta \\
&= \int_0^{2\pi} e^{2\theta} \sqrt{5} d\theta \\
&= \left[\sqrt{5} \frac{e^{2\theta}}{2} \right]_0^{2\pi} \\
&= \frac{e^{4\pi} \sqrt{5}}{2} - \frac{\sqrt{5}}{2}
\end{aligned}$$

If any errors are found, please contact me at alvin.lin.dev@gmail.com