

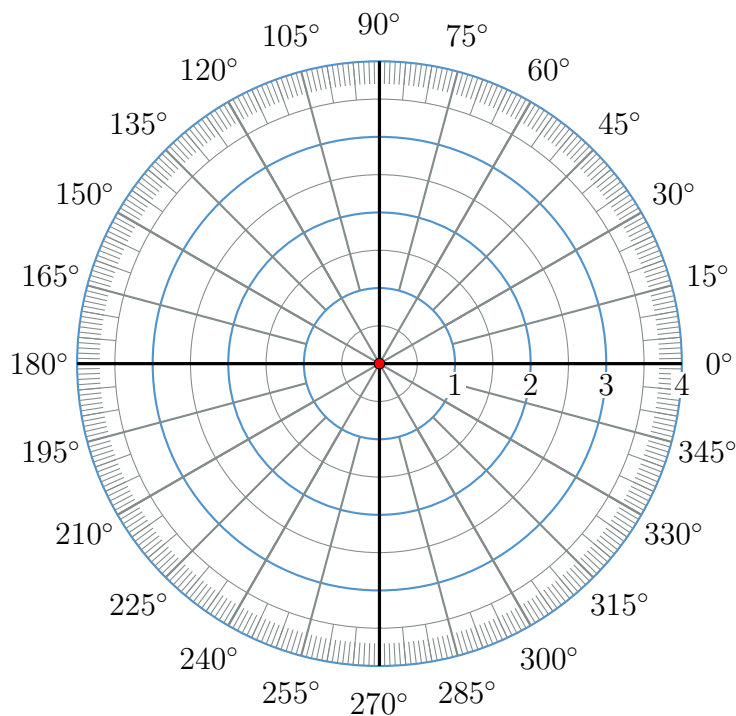
Polar Coordinates

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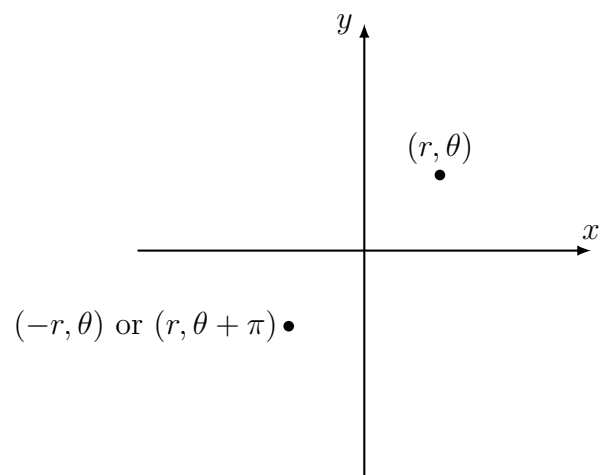
Calculus II: August 2016 - December 2016

Polar Coordinates

Instead of an x, y coordinate, polar coordinates are defined by an angle and a radius.

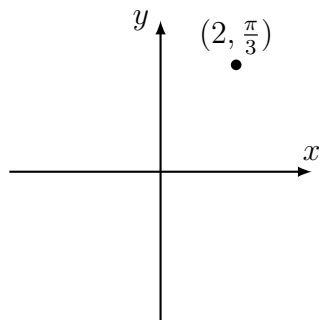


$$x = r \cos(\theta) \quad y = r \sin(\theta)$$



Problem 1a

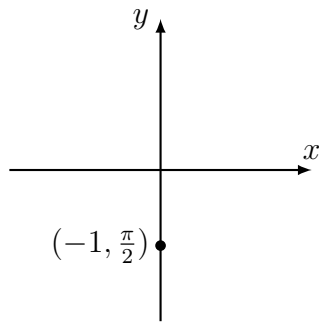
Give two other representations of $(2, \frac{\pi}{3})$.



$$(2, \frac{\pi}{3}) = (-2, \frac{4\pi}{3}) = (2, \frac{7\pi}{3})$$

Problem 1b

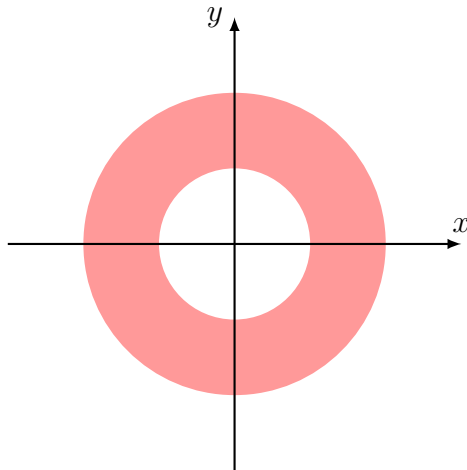
Give two other representations of $(-1, \frac{\pi}{2})$.



$$\left(-1, \frac{\pi}{2}\right) = \left(-1, \frac{5\pi}{2}\right) = \left(1, \frac{3\pi}{2}\right)$$

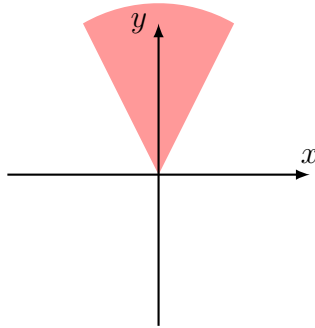
Problem 7

Shade the region $1 \leq r \leq 2$.



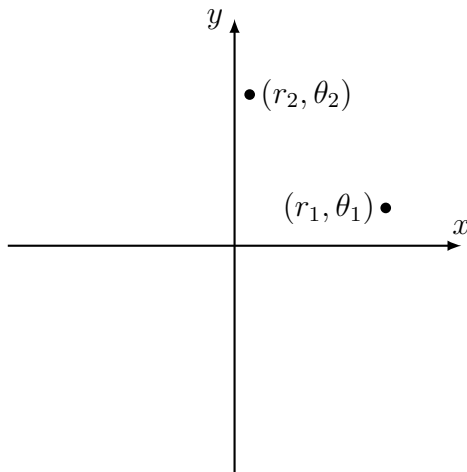
Problem 8

Shade the region $r \geq 0; \frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3}$.



Problem 14

Find the distance between the two points:



$$\begin{array}{ll}
 P_1(x_1, y_1) & P_2(x_2, y_2) \\
 x_1 = r_1 \cos(\theta_1) & x_2 = r_2 \sin(\theta_2) \\
 y_1 = r_1 \sin(\theta_1) & y_2 = r_2 \sin(\theta_2)
 \end{array}$$

$$\begin{aligned}
 P_1P_2 &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
 &= \sqrt{(r_2 \cos(\theta_2) - r_1 \cos(\theta_1))^2 + (r_2 \sin(\theta_2) - r_1 \sin(\theta_1))^2} \\
 &= \sqrt{(r_2)^2 \cos^2(\theta_2) + (r_1)^2 \cos^2(\theta_1) + (r_2)^2 \sin^2 \theta_2 + (r_1)^2 \sin^2(\theta_1) - 2r_1r_2 \sin(\theta_1) \sin(\theta_2)} \\
 &= \sqrt{(r_1)^2 + (r_2)^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)} \\
 &\text{(Law of Cosines)}
 \end{aligned}$$

We could also have drawn a triangle and applied the Law of Cosines, but this shows a the derivation of the law.

Problem 21

Convert the equation to polar coordinates:

$$x = 3$$

$$r \cos(\theta) = 3$$

Problem 22

Convert the equation to polar coordinates:

$$x^2 + y^2 = 9$$

$$(r \cos(\theta))^2 + (r \sin(\theta))^2 = 9$$

$$r^2(\cos^2(\theta) + \sin^2(\theta)) = 9$$

$$r^2 = 9$$

$$r = 3$$

Example 1

Find the slope of the tangent line of $r = 2 - \sin(\theta)$ at the point $\theta = \frac{\pi}{3}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\ x &= r \cos(\theta) \\ &= (2 - \sin(\theta)) \cos(\theta) \\ \frac{dx}{d\theta} &= (2 - \sin(\theta))(-\sin(\theta)) + (\cos(\theta))(-\cos(\theta)) \\ &= -2\sin(\theta) - \cos(2\theta) \\ y &= r \sin(\theta) \\ &= (2 - \sin(\theta)) \sin(\theta) \\ &= 2\sin(\theta) - \sin^2(\theta) \\ \frac{dy}{d\theta} &= 2\cos(\theta) - 2\sin(\theta)\cos(\theta) \\ &= 2\cos(\theta) - \sin(2\theta) \\ \frac{dy}{dx} &= \frac{2\cos(\theta) - \sin(2\theta)}{-2\sin(\theta) - \cos(2\theta)} \\ &= \frac{2\cos(\frac{\pi}{3}) - \sin(\frac{2\pi}{3})}{-2\sin(\frac{\pi}{3}) - \cos(\frac{2\pi}{3})}\end{aligned}$$

Example 2

Finding the horizontal and vertical tangents:

$$r = 1 + \cos(\theta)$$

Horizontal Tangent:

$$\frac{dy}{d\theta} = 0$$

$$y = r \sin(\theta)$$

$$y = (1 + \cos(\theta)) \sin(\theta)$$

$$\begin{aligned}\frac{dy}{d\theta} &= -\sin^2(\theta) + (1 + \cos(\theta)) \cos(\theta) \\ &= \cos(\theta) + \cos^2(\theta) - \sin^2(\theta) \\ &= \cos^2(\theta) - (1 - \cos^2(\theta)) + \cos(\theta) = 0\end{aligned}$$

$$\cos^2(\theta) - (1 - \cos^2(\theta)) + \cos(\theta) = 0$$

$$\cos^2(\theta) - 1 + \cos^2(\theta) + \cos(\theta) = 0$$

$$2 \cos^2(\theta) + \cos(\theta) + 1 = 0$$

$$2 \cos^2(\theta) + 2 \cos(\theta) - \cos(\theta) - 1 = 0$$

$$2 \cos(\theta)(\cos(\theta) + 1) - (\cos(\theta) + 1) = 0$$

$$(\cos(\theta) + 1)(2 \cos(\theta) - 1) = 0$$

$$\cos(\theta) = \frac{1}{2} \quad \cos(\theta) = -1$$

$$\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

Points of tangency:

$$\begin{aligned}(r, \theta) &= \left(\frac{3}{2}, \frac{\pi}{3}\right) \\ &= (0, \pi) \\ &= \left(\frac{3}{2}, \frac{5\pi}{3}\right)\end{aligned}$$

Vertical Tangent:

$$\frac{dx}{d\theta} = 0$$

$$x = r \cos(\theta)$$

$$= (1 + \cos(\theta)) \cos(\theta)$$

$$r = \cos(\theta) + \cos^2(\theta)$$

$$\frac{dx}{d\theta} = -\sin(\theta) + 2 \cos(\theta)(-2 \sin(\theta))$$

$$= -\sin(\theta)(1 + 2 \cos(\theta)) = 0$$

$$-\sin(\theta)(1 + 2\cos(\theta)) = 0$$

$$\sin(\theta) = 0 \quad \cos(\theta) = -\frac{1}{2}$$

$$\theta = 0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}$$

Points of tangency:

$$(r, \theta) = \left(\frac{1}{2}, \frac{2\pi}{3}\right)$$

$$= \left(\frac{1}{2}, \frac{4\pi}{3}\right)$$

$$= (2, 0)$$

$$= (0, \pi) \leftarrow \text{Undefined at this point}$$

If any errors are found, please contact me at alvin.lin.dev@gmail.com