Calculus With Parametric Curves

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Calculus With Parametric Curves

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$$

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}t}{\mathrm{d}x}\\ &= \frac{\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathrm{d}y}{\mathrm{d}x})}{\frac{\mathrm{d}x}{\mathrm{d}t}}\end{aligned}$$

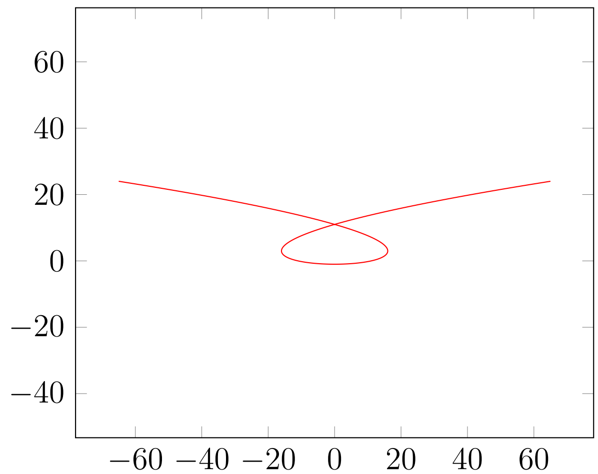
Using this, we can write the arc length function as:

$$\begin{aligned}L &= \int_a^b \sqrt{1 + (\frac{\mathrm{d}y}{\mathrm{d}x})^2}\mathrm{d}x\\ &= \int_a^b \sqrt{1 + (\frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t})^2}\mathrm{d}x\\ &= \int_a^b \frac{\sqrt{(\mathrm{d}x/\mathrm{d}t)^2 + (\mathrm{d}y/\mathrm{d}t)^2}}{\mathrm{d}x/\mathrm{d}t}\mathrm{d}x\\ &= \int_a^b \sqrt{(\frac{\mathrm{d}x}{\mathrm{d}t})^2 + (\frac{\mathrm{d}y}{\mathrm{d}t})^2}\mathrm{d}t\end{aligned}$$

Example 1

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$:

$$x = t^3 - 12t \quad y = t^2 - 1$$



$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2 - 12 = 3(t^2 - 4)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2t$$

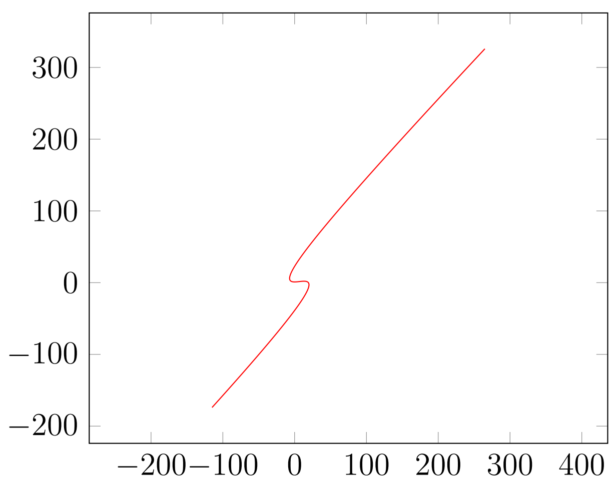
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}x}{\mathrm{d}t}}{\frac{\mathrm{d}y}{\mathrm{d}t}} = \frac{2t}{3t^2 - 12}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathrm{d}y}{\mathrm{d}x}) &= \frac{(3t^2-12)(2)-2t(6t)}{(3t^2-12)^2} \\
&= \frac{6t^2-24-12t^2}{(3t^2-12)^2} \\
&= \frac{-6(t^2+4)}{9(t^2-4)^2} \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathrm{d}y}{\mathrm{d}x})}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\
&= \frac{2t}{3t^2-12} \times \frac{1}{3(t^2-4)} \\
&= \frac{-6(t^2+4)}{27(t^2-4)^3} \\
&= -\frac{2}{9}\frac{(t^2+4)}{(t^2-4)^3}
\end{aligned}$$

Example 2

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$x = 2t^3 + 3t^2 - 12t \quad y = 2t^3 + 3t^2 + 1$$

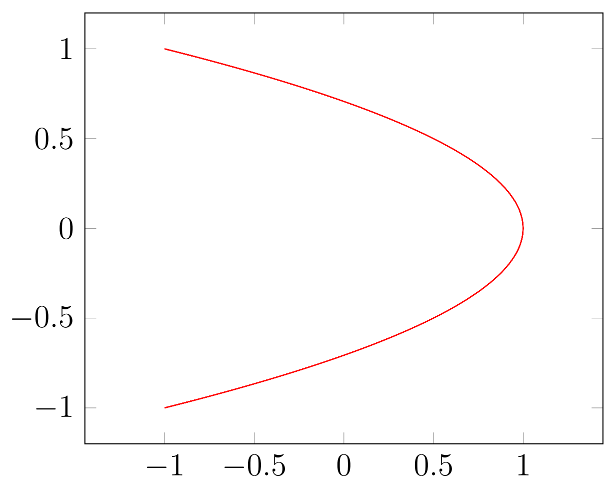


$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= 6t^2 + 6t - 12 \\ &= 6(t^2 + t - 2) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 6t^2 + 6t \\ &= 6t(t+1) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{6t(t+1)}{t(t^2+t-2)} \\ &= \frac{t(t+1)}{(t+2)(t-2)}\end{aligned}$$

Example 3

Find the points at which the function is convex:

$$x = \cos(2t) \quad y = \cos(t) \quad 0 < t < \pi$$



$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= -2\sin(2t) \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= -\sin^2(t) \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\
&= \frac{-\sin(t)}{-2\sin(2t)} \\
&= \frac{\sin(t)}{2 \times 2\sin(t)\cos(t)} \\
&= \frac{\sec(t)}{4} \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathrm{d}y}{\mathrm{d}x})}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\
&= \frac{\sec(t)\tan(t)}{4} \times \frac{1}{-2\sin(2t)} \\
&= -\frac{\sec(t)\tan(t)}{8\sin(2t)} \\
&= -\frac{\sin(t)}{8\cos(t)\cos(t)2\sin(t)\cos(t)} \\
&= -\frac{\sec^3(t)}{16}
\end{aligned}
$$

The function is convex wherever $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$:

$$-\frac{\sec^3(t)}{16} > 0$$

$$\sec^3(t) < 0$$

$$\sec(t) < 0$$

$$\cos(t) < 0$$

$$\frac{\pi}{2} < t < \frac{3\pi}{2}$$

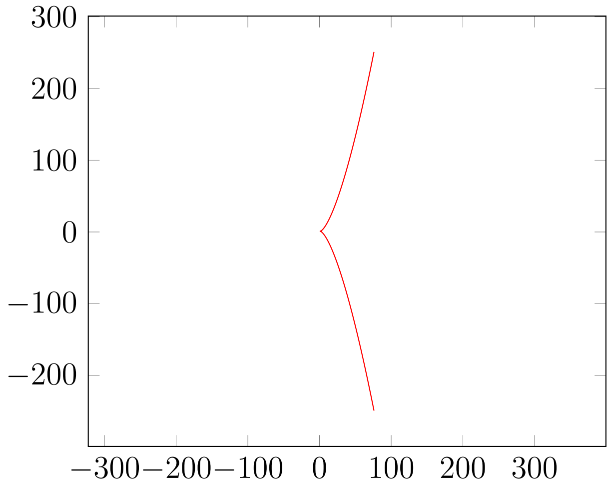
Since we were given the precondition $0 < t < \pi$, we only care about the interval:

$$\frac{\pi}{2} < t < \pi$$

Example 4

Find the equation of the tangent line at (4,3):

$$x = 3t^2 + 1 \quad y = 2t^3 + 1$$



The equation of the tangent line is of the form:

$$y - y_1 = m(x - x_1) \text{ where } m = \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= 6t \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= 6t^2 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\
&= \frac{6t^2}{6t} \\
&= t
\end{aligned}$$

Tangent line at (4,3):

$$\begin{aligned}
y - y_1 &= \frac{\mathrm{d}y}{\mathrm{d}x}(x - x_1) \\
y - y_1 &= t(x - x_1) \\
3 - (2t^3 + 1) &= t(4 - (3t^2 + 1)) \\
3 - 2t^3 - 1 &= y(4 - 3t^2 - 1) \\
2 - 2t^3 &= 4t - 3t^3 - t
\end{aligned}$$

$$\begin{aligned}
t^3 - 3t + 2 &= 0 \\
t^3 - t^2 + t^2 - t - 2t + 2 &= 0 \\
t^2(t - 1) + t(t - 1) - 2(t - 1) &= 0 \\
(t - 1)(t^2 + t - 2) &= 0 \\
(t - 1)(t + 2)(t - 1) &= 0 \\
(t - 1)^2(t - 2) &= 0
\end{aligned}$$

$$t = 1 \quad t = 2$$

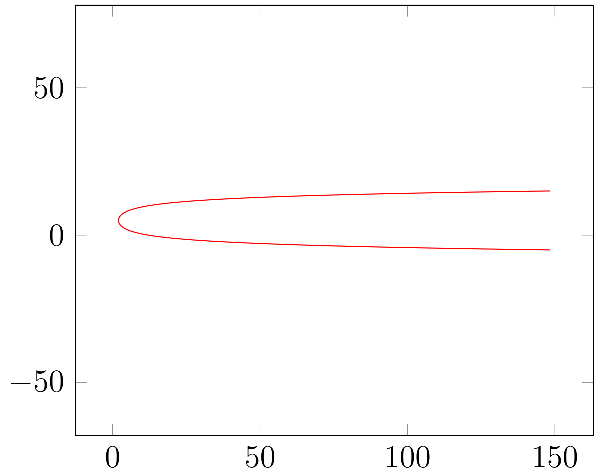
At $t = 1$, $(x, y) = (4, 3)$, therefore the equation of the tangent line is:

$$\begin{aligned}
y - 3 &= 1(x - 4) \\
y &= x - 1
\end{aligned}$$

Example 5

Find the arc length of the curve:

$$x = \mathrm{e}^t + \mathrm{e}^{-t} \quad y = 5 - 2t \quad 0 \leq t \leq 3$$

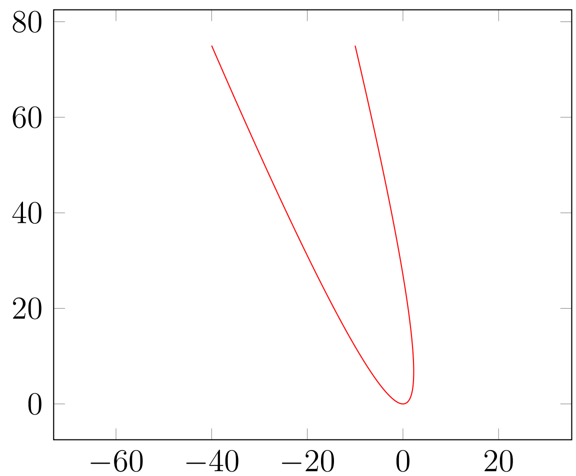


$$
\begin{aligned}
L &= \int_a^b \sqrt{1+(f'(x))^2}\mathrm{d}x \\
f'(x) &= \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} \\
L &= \int_a^b \sqrt{(\frac{\mathrm{d}x}{\mathrm{d}t})^2 + (\frac{\mathrm{d}y}{\mathrm{d}t})^2}\mathrm{d}t \\
&= \int_0^3 \sqrt{(\mathrm{e}^t - \mathrm{e}_{-t})^2 + (-2)^2}\mathrm{d}t \\
&= \int_0^3 \sqrt{\mathrm{e}^{2t} + \mathrm{e}^{-2t} - 2\mathrm{e}^t\mathrm{e}^{-t} + 4}\mathrm{d}t \\
&= \int_0^3 \sqrt{\mathrm{e}^{2t} + \mathrm{e}^{-2t} + 2} \\
&= \int_0^3 \sqrt{(\mathrm{e}^t + \mathrm{e}^{-t})^2}\mathrm{d}t \\
&= \int_0^3 (\mathrm{e}^t + \mathrm{e}^{-t})\mathrm{d}t \\
&= \left[\mathrm{e}^t - \mathrm{e}^{-t}\right]_0^3 \\
&= \mathrm{e}^3 - \mathrm{e}^{-3}
\end{aligned}
$$

Example 6

Find the surface area of the solid of revolution:

$$x = 3t - t^2 \quad y = 3t^2 \quad 0 \leq t \leq 1$$



$$
\begin{aligned}
S &= \int 2\pi y \mathrm{d}S \\
&= \int_a^b 2\pi y \sqrt{1 + (\frac{\mathrm{d}y}{\mathrm{d}x})^2} \mathrm{d}x \\
&= 2\pi \int_a^b y \sqrt{(\frac{\mathrm{d}x}{\mathrm{d}t})^2 + (\frac{\mathrm{d}y}{\mathrm{d}t})^2} \mathrm{d}t \\
&= 2\pi \int_0^1 (3t^2) \sqrt{(3 - 2t^2)^2 + (6t)^2} \mathrm{d}t \\
&= 6\pi \int_0^1 t^2 \sqrt{9 + 9t^4 - 12t^2 + 36t^2} \mathrm{d}t \\
&= 6\pi \int_0^1 t^2 \sqrt{3^2 + (3t^2)^2 + 2(3)(3t^2)} \mathrm{d}t \\
&= 6\pi \int_0^1 t^2 \sqrt{(3 + 3t^2)^2} \mathrm{d}t \\
&= 6\pi \int_0^1 t^2 (3 + 3t^2) \mathrm{d}t \\
&= 18\pi \int_0^1 (t^2 + t^4) \mathrm{d}t \\
&= 18\pi \left[\frac{t^3}{3} + \frac{t^5}{5} \right]_0^1 \\
&= 18\pi (\frac{1}{3} + \frac{1}{5}) \\
&= 18\pi (\frac{8}{15}) \\
&= \frac{48\pi}{5}
\end{aligned}
$$

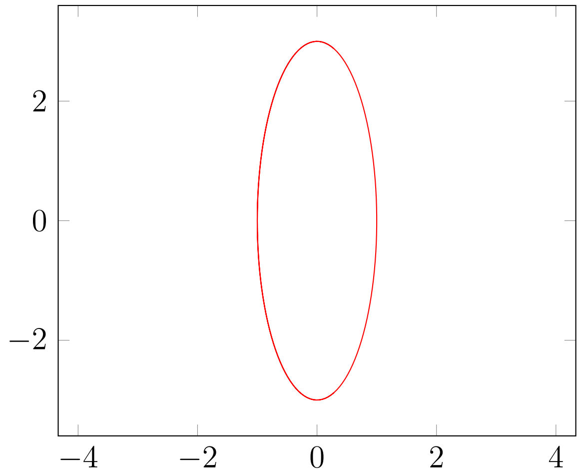
Example 7

$$x = a\cos(\theta) \quad y = b\sin(\theta) \quad 0 \leq \theta \leq 2\pi$$

$$\cos(\theta) = \frac{x}{a}$$

$$\sin(\theta) = \frac{y}{b}$$

$$\sin^2(\theta) + \cos^2(\theta) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



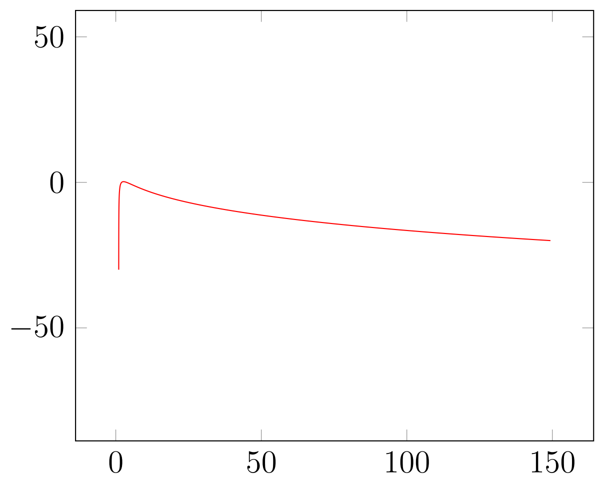
$$
\begin{aligned}
A &= 4\int_a^b y\mathrm{d}x \\
&= 4\int_{\pi/2}^{0} (b\sin(\theta))(-a\sin(\theta))\mathrm{d}\theta \\
&= -4ab\int_{\pi/2}^{0} \sin^2(\theta)\mathrm{d}\theta \\
&= 4ab\int_0^{\pi/2} \frac{1}{2} - \frac{1}{2}\cos(2\theta)\mathrm{d}theta \\
&= 2ab\int_0^{\pi/2} 1 - \cos(2\theta)\mathrm{d}\theta \\
&= 2ab\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\pi/2} \\
&= 2ab\left[(\frac{\pi}{2} - \frac{\sin(\pi)}{2}) - (0 - \frac{\sin(\theta)}{2})\right] \\
&= ab\pi
\end{aligned}
$$

Problem 33

Find the area under the curve bounded by:

$$x = 1 + \mathrm{e}^t \quad y = t - t^2$$

and the x-axis:



$$A = \int_a^b y \mathrm{d}x$$

$$y = 0 = t - t^2$$

$$t(1-t) = 0$$

$$t = 0 \quad (x = 2) \qquad t = 1 \quad (x = 1 + \mathrm{e})$$

$$\begin{aligned} A &= \int_2^{1+\mathrm{e}} y \mathrm{d}x \\ &= \int_{x=2}^{x=1+\mathrm{e}} (t - t^2) \mathrm{d}x \\ &= \int_0^1 (t - t^2)\mathrm{e}^t \mathrm{d}t \\ &= (t - t^2)\mathrm{e}^t - \int (1 - 2t)\mathrm{e}^t \mathrm{d}t \quad \text{(Integration By Parts)} \\ &= t(1-t)\mathrm{e}^t - \left[(1-2t)\mathrm{e}^t - \int -2\mathrm{e}^t \mathrm{d}t\right] \\ \int_0^1 (t - t^2)\mathrm{e}^t \mathrm{d}t &= \Big[t(1-t)\mathrm{e}^t - (1-2t)\mathrm{e}^t - 2\mathrm{e}^t\Big]_0^1 \\ &= 3 - \mathrm{e} \end{aligned}$$

If any errors are found, please contact me at alvin.lin.dev@gmail.com