

Improper Integrals

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Improper Integrals

$$\int_0^1 x dx$$

This is a proper integral. It represents the area under the curve of the function $y = x$ from 0 to 1.

$$\int_0^\infty x dx$$

This is an example of an improper integral. It has no proper meaning because you cannot take the area under the curve from 0 to ∞ . For any integral of this form, the problem is only as hard as is the integration of $f(x)$. We can rewrite it as:

$$\lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

Practice Problem 5

$$\int_3^\infty \frac{1}{(x-2)^{3/2}} dx$$

$$\lim_{b \rightarrow \infty} \int_3^b \frac{1}{(x-2)^{3/2}} dx$$

$$\lim_{b \rightarrow \infty} \int_3^b (x-2)^{-3/2} dx$$

Solve it as an indefinite integral:

$$\int (x-2)^{-3/2} dx$$

$$\text{Let : } f'(x) = x^{-3/2} dx \quad g(x) = x - 2$$

$$f(x) = \frac{-2}{\sqrt{x}} \quad g'(x) = dx$$

$$f(g(x)) = \int (x - 2)^{-3/2} dx = -\frac{2}{\sqrt{x - 1}} + C$$

Evaluate using the original limits:

$$\begin{aligned} & -2 \lim_{b \rightarrow \infty} \left[\frac{1}{\sqrt{x - 1}} dx \right]_3^b \\ & -2 \lim_{b \rightarrow \infty} \left[\frac{1}{\sqrt{b - 2}} - \frac{1}{\sqrt{3 - 2}} \right] \\ & \quad -2(0 - 1) \\ & \quad = -2 \end{aligned}$$

Practice Problem 11

$$\int_0^{\infty} \frac{x^2}{\sqrt{1 + x^2}} dx$$

$$\lim_{b \rightarrow \infty} \int_0^b \frac{x^2}{\sqrt{1 + x^2}} dx$$

Solve it as an indefinite integral:

$$\int \frac{x^2}{\sqrt{1 + x^2}} dx$$

$$\text{Let : } t = 1 + x^2$$

$$dt = 3x^2 dx$$

$$\int \frac{x^3}{\sqrt{t} 3x^2} dt$$

$$\frac{1}{3} \int \frac{1}{\sqrt{t}} dt$$

$$\frac{2}{3} \sqrt{t}$$

$$\frac{2}{3}\sqrt{1+x^3}$$

Evaluate using the original limits:

$$\begin{aligned} & \frac{2}{3} \lim_{b \rightarrow \infty} \left[\sqrt{1+x^3} \right]_0^b \\ & \frac{2}{3} \lim_{b \rightarrow \infty} \left[\sqrt{1+b^3} - \sqrt{1+0^3} \right] \\ & = \infty \end{aligned}$$

Practice Problem 21

$$\begin{aligned} & \int_0^\infty \frac{\ln(x)}{x} dx \\ & \lim_{b \rightarrow \infty} \int_1^b \frac{\ln(x)}{x} dx \end{aligned}$$

Solve it as an indefinite integral:

$$\begin{aligned} & \int \frac{\ln(x)}{x} dx \\ \text{Let } & t = \ln(x) \\ & dt = \frac{1}{x} dx \\ & \int \frac{t}{x} x dt \\ & \int t dt \\ & \frac{1}{2} t^2 \\ & \frac{(\ln(x))^2}{2} \end{aligned}$$

Evaluate using the original limits:

$$\begin{aligned} & \lim_{b \rightarrow \infty} \left[\frac{(\ln(x))^2}{2} \right]_1^b \\ & \lim_{b \rightarrow \infty} \left[\frac{(\ln(b))^2}{2} - \frac{(\ln(1))^2}{2} \right] \\ & = \infty \end{aligned}$$

Review

$$\int_{-\infty}^{\infty} \frac{x^2}{9+x^6} dx$$
$$\int_{-\infty}^0 \frac{x^2}{9+x^6} dx + \int_0^{\infty} \frac{x^2}{9+x^6} dx$$

Because $\frac{x^2}{9+x^6}$ is an even function, we can rewrite this as:

$$2 \int_0^{\infty} \frac{x^2}{9+x^6} dx$$

Solve it as an indefinite integral:

$$2 \int \frac{x^2}{9+x^6} dx$$
$$2 \int \frac{x^2}{9+(x^3)^2} dx$$
$$\text{Let : } 3t = x^3$$
$$3dt = 3x^2 dx$$
$$dt = x^2 dx$$
$$2 \int \frac{x^2}{9+(3t)^2} x^2 dt$$
$$2 \int \frac{1}{9+9t^2} dt$$
$$\frac{2}{9} \int \frac{1}{1+t^2} dt$$
$$\frac{2}{9} \tan^{-1}(t) + C$$
$$\frac{2}{9} \tan^{-1}\left(\frac{x^3}{3}\right) + C$$

Evaluate using the original limits:

$$\frac{2}{9} \lim_{a \rightarrow \infty} \left[\tan^{-1}\left(\frac{x^3}{3}\right) \right]_0^a$$

$$\begin{aligned} \frac{2}{9} \lim_{a \rightarrow \infty} \left[\tan^{-1}\left(\frac{a^3}{3}\right) - \frac{0^3}{3} \right] \\ \frac{2}{9} \lim_{a \rightarrow \infty} \left[\frac{\pi}{2} - 0 \right] \\ = \frac{\pi}{9} \end{aligned}$$

Review 2

$$\int_1^{\infty} \frac{1}{x^{1/2} + x^{3/2}} dx$$

$$\int_1^{\infty} \frac{1}{\sqrt{x}(1+x)} dx$$

Solve it as an indefinite integral:

$$\int \frac{1}{\sqrt{x}(1+x)} dx$$

Let : $x = t^2$

$$dx = 2t dt$$

$$\int \frac{1}{t(1+t^2)} 2t dt$$

$$2 \int \frac{1}{1+t^2} dt$$

$$2 \tan^{-1}(t) = 2 \tan^{-1}(\sqrt{x})$$

Evaluate using the original limits:

$$\lim_{a \rightarrow \infty} \left[2 \tan^{-1}(\sqrt{x}) \right]_1^a$$

$$\lim_{a \rightarrow \infty} \left[2 \tan^{-1}(\sqrt{a}) - 2 \tan^{-1}(1) \right]_1^a$$

$$2\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{2}$$

Review 3

$$\begin{aligned}
 & \int \frac{x+4}{x^2+2x+5} dx \\
 & \int \frac{x+4}{x^2+2x+5} dx \\
 & \int \frac{x+4}{x^2+2x+5} dx \\
 & \frac{1}{2} \int \frac{2x+8}{x^2+2x+5} dx \\
 & \frac{1}{2} \int \frac{2x+2+6}{x^2+2x+5} dx \\
 & \frac{1}{2} \int \frac{2x+2}{x^2+2x+5} + \frac{6}{x^2+2x+5} dx \\
 & \frac{1}{2} \ln|x^2+2x+5| + C + \frac{1}{2} \int \frac{6}{x^2+2x+5} dx \\
 & \frac{1}{2} \int \frac{6}{x^2+2x+5} dx = 3 \int \frac{1}{x^2+2x+1+4} dx \\
 & 3 \int \frac{1}{(x+1)^2+4} dx \\
 & \text{Let : } x+1 = 2 \tan(\theta) \\
 & dx = 2 \sec^2(\theta) d\theta \\
 & 3 \int \frac{1}{(2 \tan(\theta))^2+4} 2 \sec^2(\theta) d\theta \\
 & 3 \int \frac{2 \sec^2(\theta)}{4(\tan^2(\theta)+1)} d\theta \\
 & \frac{3}{2} \int \frac{\sec^2(\theta)}{\sec^2(\theta)} d\theta \\
 & \frac{3}{2} \int 1 d\theta \\
 & \frac{3}{2} \theta \\
 & x+1 = 2 \tan(\theta) \quad \theta = \tan^{-1}\left(\frac{x+1}{2}\right) \\
 & = \frac{3}{2} \tan^{-1}\left(\frac{x+1}{2}\right)
 \end{aligned}$$

If any errors are found, please contact me at alvin.lin.dev@gmail.com