

Approximate Integration

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Approximate Integration

$$\int_0^1 e^{x^2} dx$$

Normally we would solve this problem by integrating $\int e^{x^2} = F(x)$ and then using the Fundamental Theorem of Calculus ($F(1) - F(0)$). However, there is no known way to integrate e^{x^2} (or at least it becomes very difficult). We can find a numerical approximation of it instead.

$$\int_a^b f(x)dx = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n f(x_i^*)\Delta x$$

$$x_i^* \in [x_{i-1}, x_i]$$

$$\Delta x = \frac{b-a}{n}$$

When we integrate, we are taking the sum of areas of rectangles under the curve as the width of the rectangle approaches 0. This gives us an exact value. If we cannot do that, then we can approximate the integral by removing the limit.

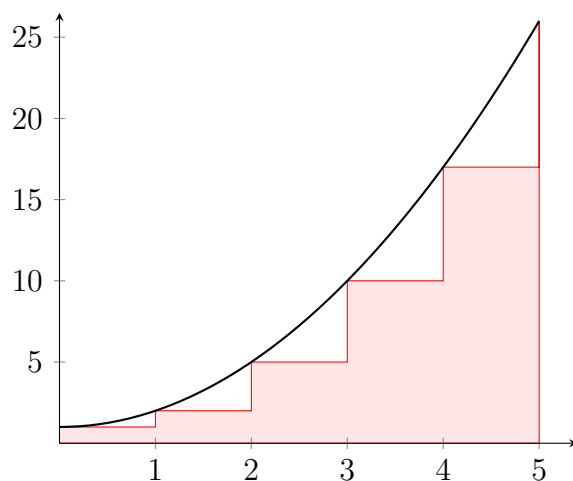
$$\int_a^b f(x)dx \approx \sum_{i=1}^n f(x_i^*)\Delta x$$

This splits the integral into n discrete points, each of which we can calculate a rectangular area from.

Left Riemann Sum Approximation

$$\begin{aligned} L_n &= \sum_{i=1}^n f(x_{i-1})\Delta x \\ &= \Delta x \left[f(x_0) + f(x_1) + \dots + f(x_{n-1}) \right] \end{aligned}$$

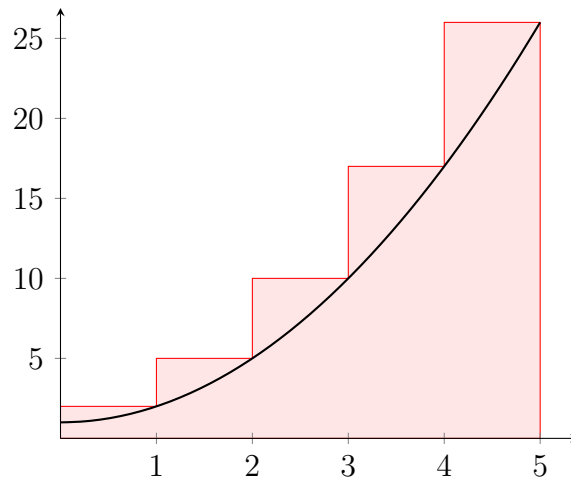
If we split the area under the curve into n sections, we can use $f(x_i)$ as the height of **the left side of a rectangle** under the curve. By calculating the sum of the areas of the n rectangles starting from x_0 , we achieve the left Riemann sum approximation of the integral.



Right Riemann Sum Approximation

$$\begin{aligned} R_n &= \sum_{i=1}^n (f(x_i)\Delta x) \\ &= \Delta x \left[f(x_1) + f(x_2) + \dots + f(x_n) \right] \end{aligned}$$

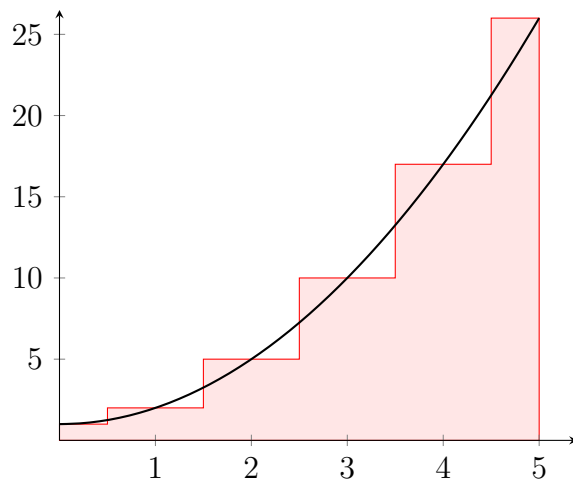
If we split the area under the curve into n sections, we can use $f(x_i)$ as the height of **the right side of a rectangle** under the curve. By calculating the sum of the areas of the n rectangles starting from x_1 , we achieve the right Riemann sum approximation of the integral.



Midpoint Approximation

$$\begin{aligned}
 M_n &= \sum_{i=1}^n f(\bar{x}_i) \Delta x \\
 &= \Delta x \left[f(\bar{x}_1) + f(\bar{x}_2) + \dots + f(\bar{x}_n) \right] \\
 &\text{where } \bar{x}_i = \frac{x_{i-1} + x_i}{2}
 \end{aligned}$$

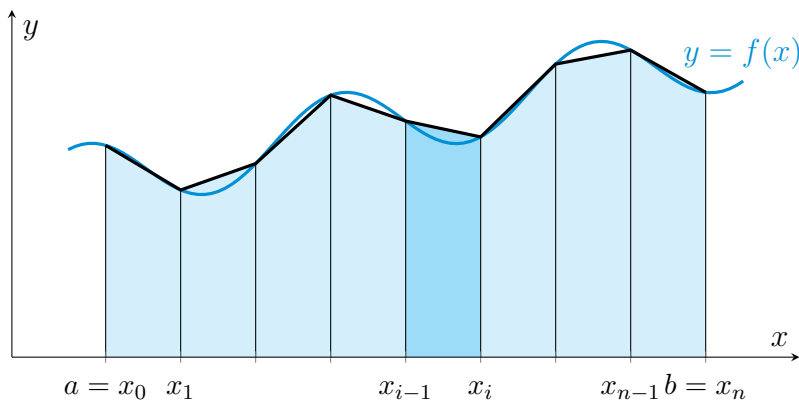
If we split the area under the curve into n sections, we can calculate the midpoints between each section \bar{x}_i . We can use $f(\bar{x}_i)$ as the heights of our rectangles and by summing up the areas of the rectangles, we achieve the midpoint approximation of the integral.



Trapezoidal Rule Approximation

$$T_n = \frac{\Delta x}{2} \left[f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n) \right]$$

If we split the area under the curve into n sections, we can use $f(x_i)$ as bases of trapezoids. By summing up the areas of the trapezoids, we achieve the trapezoidal approximation of the integral.



Simpson's Rule

$$S_n = \frac{\Delta x}{3} \left[f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]$$

Simpson's rule tries to get a parabolic approximation of the integral at every n^{th} segment.

No visualization

Example

$$\begin{aligned}\int_0^2 \frac{x}{1+x^2} dx &= \left[\frac{1}{2} \ln |1+x^2| \right]_0^2 \\ &= \frac{1}{2} \ln(5) - 0 \\ &\approx 0.804719\end{aligned}$$

If we use the left Riemann sum approximation with $n = 10$:

$$\begin{aligned}L_n &= \Delta x \left[f(x_1) + f(x_2) + \dots + f(x_n) \right] \\ \Delta x &= \frac{2-0}{10} \\ &= \frac{2}{10} \left[f(0) + f\left(\frac{2}{10}\right) + \dots + f\left(\frac{18}{10}\right) \right] \\ &\approx 0.708985\dots\end{aligned}$$

If we use the right Riemann sum approximation with $n = 10$:

$$\begin{aligned}R_n &= \Delta x \left[f(x_1) + f(x_2) + \dots + f(x_n) \right] \\ \Delta x &= \frac{2-0}{10} \\ &= \frac{2}{10} \left[f\left(\frac{2}{10}\right) + f\left(\frac{4}{10}\right) + \dots + f(2) \right] \\ &\approx 0.798391\dots\end{aligned}$$

If we use the midpoint sum approximation with $n = 10$:

$$M_n = \Delta x \left[f(\bar{x}_1) + f(\bar{x}_2) + \dots + f(\bar{x}_n) \right]$$

$$\begin{aligned}\Delta x &= \frac{2-0}{10} \\ &= \frac{2}{10} \left[f\left(\frac{1}{10}\right) + f\left(\frac{3}{10}\right) + \dots + f\left(\frac{19}{20}\right) \right] \\ &\approx 0.715920\dots\end{aligned}$$

If we use the trapezoidal rule approximation with $n = 10$:

$$T_n = \frac{\Delta x}{2} \left[f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n) \right]$$

$$\begin{aligned}\Delta x &= \frac{2-0}{10} \\ &= \frac{2}{10} \times \frac{1}{2} \left[f(0) + 2f\left(\frac{2}{10}\right) + \dots + 2f\left(\frac{18}{20}\right) + f(2) \right] \\ &\approx 0.753688\end{aligned}$$

If we use Simpson's Rule with $n = 10$:

$$S_n = \frac{\Delta x}{3} \left[f(x_1) + 4f(x_1) + 2f(x_2) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]$$

$$\begin{aligned}\Delta x &= \frac{2-0}{10} \\ &= \frac{2}{10} \times \frac{1}{3} \left[f(0) + 4f\left(\frac{2}{10}\right) + 2f\left(\frac{4}{10}\right) \dots + 4f\left(\frac{18}{20}\right) + f(2) \right] \\ &\approx 0.716586\end{aligned}$$

If any errors are found, please contact me at alvin.lin.dev@gmail.com