

Trigonometric Substitution

Alvin Lin

Calculus II: August 2016 - December 2016

Trigonometric Substitution

$$\int \sin^4(x) \cos^3(x) dx$$

When you have a product of sin and cos of different powers, you have three different possibilities:

- They are both even powers.
- They are both odd powers.
- One exponent is odd and the other is even.

We can rewrite this problem as:

$$\int \sin^4(x) \cos^2(x) \cos(x) dx$$

We want even powers of

$$\int \sin^4(x) (1 - \sin^2(x))^2(x) \cos(x) dx$$

Now we can use substitution:

$$\text{Let : } \sin(x) = t$$

$$\cos(x) = \frac{dt}{dx} \quad \cos(x) dx = dt$$

$$\int t^4(1-t^2)dt$$

$$\int t^4 - t^6 dt = \int t^4 dt - \int t^6 dt$$

$$\frac{t^5}{5} - \frac{t^7}{7} + C$$

$$\frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C$$

And in the wise words of Professor Khan: "These terms are like Hillary and Trump supporters and we cannot combine them."

Another case is where we have a difficult term inside a radical in the denominator.

$$\int \frac{dx}{x^2\sqrt{4-x^2}}$$

Note that the term in the radical has a form similar to the trigonometric identities above.

$$\text{Let : } x = 2 \sin(\theta)$$

$$dx = 2 \cos(\theta)d\theta$$

$$\int \frac{2 \cos(\theta)d\theta}{(2 \sin(\theta))^2\sqrt{4-4 \sin^2(\theta)}}$$

By substituting for $2 \sin(\theta)$, we can turn the radical into the form of a trigonometric identity.

$$\int \frac{2 \cos(\theta)}{4 \sin^2(\theta)\sqrt{4}\sqrt{1-1 \sin^2(\theta)}}d\theta$$

In this case, we are using the identity $\sin^2(\theta) + \cos^2(\theta) = 1$ which we can rewrite as $\cos^2(\theta) = 1 - \sin^2(\theta)$.

$$\frac{1}{4} \int \frac{\cos \theta}{\sin^2(\theta)\sqrt{\cos^2(\theta)}}d\theta$$

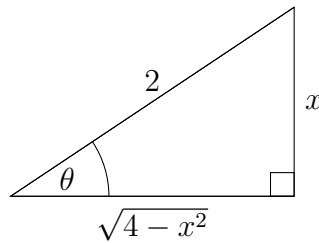
$$\frac{1}{4} \int \frac{\cos \theta}{\sin^2(\theta)\cos(\theta)}d\theta$$

$$\frac{1}{4} \int \frac{1}{\sin^2(\theta)}d\theta$$

$$\frac{1}{4} \int \csc^2(\theta) d\theta$$

$$\frac{1}{4} \cot(\theta) + C$$

To substitute back, we must imagine a triangle with angle θ . Given our first substitution $x = 2 \sin(\theta)$, we can rewrite it as $\sin(\theta) = \frac{x}{2} = \frac{\text{opp}}{\text{hyp}}$. If our triangle has opposite side x and hypotenuse 2, then the adjacent side must be $\sqrt{4 - x^2}$.



Therefore: $\cot(\theta) = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{4-x^2}}{x}$

$$\frac{1}{4} \cot(\theta) + C = \frac{1}{4} \frac{\sqrt{4-x^2}}{x} + C$$

$$= \frac{\sqrt{4-x^2}}{4x} + C$$

Practice Problem 4

$$\int \frac{x^2}{\sqrt{9-x^2}} dx$$

$$\text{Let : } x = 3 \sin(\theta)$$

$$dx = 3 \cos(\theta) d\theta$$

$$\int \frac{(3 \sin(\theta))^2}{\sqrt{9 - (3 \sin(\theta))^2}} 3 \cos(\theta) d\theta$$

$$\int \frac{27 \sin^2(\theta) \cos(\theta)}{\sqrt{9 - 9 \sin^2(\theta)}} d\theta$$

$$\begin{aligned}
& \int \frac{27 \sin^2(\theta) \cos(\theta)}{\sqrt{9}\sqrt{1 - \sin^2(\theta)}} d\theta \\
& 9 \int \frac{\sin^2(\theta) \cos(\theta)}{\sqrt{\cos^2(\theta)}} d\theta \\
& 9 \int \frac{\sin^2(\theta) \cos(\theta)}{\cos(\theta)} d\theta \\
& 9 \int \sin^2(\theta) d\theta
\end{aligned}$$

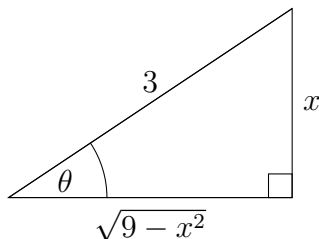
Using the double angle formulas:

$$\begin{aligned}
& 9 \int \left[\frac{1}{2} - \frac{1}{2} \cos(2\theta) \right] d\theta \\
& \frac{9}{2} \int 1 - \cos(2\theta) d\theta \\
& \frac{9}{2} \left[\theta - \frac{\sin(2\theta)}{2} \right] + C
\end{aligned}$$

Using the double angle formulas again:

$$\begin{aligned}
& \frac{9}{2} \left[\theta - \frac{2 \sin(\theta) \cos(\theta)}{2} \right] + C \\
& = \frac{9}{2} \left[\theta - \sin(\theta) \cos(\theta) \right] + C
\end{aligned}$$

Recall that we substituted $x = 3 \sin(\theta)$, which we can rewrite as $\sin(\theta) = \frac{x}{3} = \frac{\text{opp}}{\text{hyp}}$. If we imagine a triangle in which the opposite side is x and the hypotenuse is 3, then the adjacent side must be $\sqrt{9 - x^2}$.



Therefore: $\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{9-x^2}}{3}$ and $\theta = \sin^{-1}\left(\frac{x}{3}\right)$

$$\begin{aligned} \frac{9}{2} \left[\theta - \sin(\theta) \cos(\theta) \right] + C &= \frac{9}{2} \left[\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3} \frac{\sqrt{9-x^2}}{3} \right] + C \\ &= \frac{9}{2} \left[\sin^{-1}\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{9} \right] + C \end{aligned}$$

Practice Problem 6

$$\int_0^3 \frac{x}{\sqrt{36-x^2}} dx$$

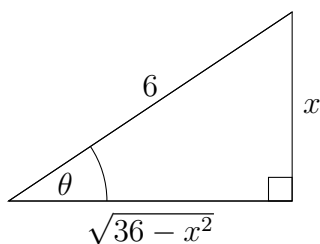
$$\text{Let : } x = 6 \sin(\theta)$$

$$dx = 6 \cos(\theta) d\theta$$

For now, we will solve the problem as an indefinite integral.

$$\begin{aligned} &\int \frac{6 \sin(\theta)}{\sqrt{36 - (6 \sin(\theta))^2}} 6 \cos(\theta) d\theta \\ &\int \frac{36 \sin(\theta) \cos(\theta)}{\sqrt{36 - 36 \sin^2(\theta)}} d\theta \\ &\int \frac{36 \sin(\theta) \cos(\theta)}{\sqrt{36} \sqrt{1 - \sin^2(\theta)}} d\theta \\ &6 \int \frac{\sin(\theta) \cos(\theta)}{\sqrt{\cos^2(\theta)}} d\theta \\ &6 \int \frac{\sin(\theta) \cos(\theta)}{\cos(\theta)} d\theta \\ &6 \int \sin(\theta) d\theta \\ &-6 \cos(\theta) + C \end{aligned}$$

Recall that we substituted $x = 6 \sin(\theta)$, which we can rewrite as $\sin(\theta) = \frac{x}{6} = \frac{\text{opp}}{\text{hyp}}$. If we imagine a triangle in which the opposite side is x and the hypotenuse is 6, then the adjacent side must be $\sqrt{36-x^2}$.



Therefore: $\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{36-x^2}}{6}$

$$\begin{aligned} -6 \cos(\theta) + C &= -6 \frac{\sqrt{36-x^2}}{6} + C \\ &= -\sqrt{36-x^2} + C \end{aligned}$$

Now we can use the original limits of the intergral to solve this.

$$\begin{aligned} &\left[-\sqrt{36-x^2} \right]_0^3 \\ &-\sqrt{36-3^2} - (-\sqrt{36-0^2}) \\ &-\sqrt{27} + 6 \\ &= 6 - 3\sqrt{3} \end{aligned}$$

Practice Problem 8

$$\int \frac{dt}{t^2 \sqrt{t^2 - 16}}$$

Note that the radical is of the form $t^2 - 16$. We cannot subsitute $\sin(\theta)$ into this since it will not satisfy the trigonometric identity.

$$\text{Let : } x = 4 \sec(\theta)$$

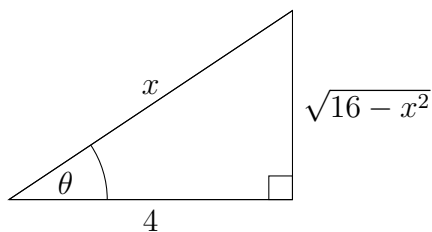
$$dx = 4 \sec(\theta) \tan(\theta) d\theta$$

$$\int \frac{4 \sec \theta \tan(\theta)}{16 \sec^2(\theta) \sqrt{(4 \sec(\theta))^2 - 16}} d(\theta)$$

$$\frac{1}{4} \int \frac{\tan(\theta)}{\sec(\theta) \sqrt{16} \sqrt{\sec^2(\theta) - 1}} d(\theta)$$

$$\begin{aligned} & \frac{1}{16} \int \frac{\tan(\theta)}{\sec(\theta)\sqrt{\tan^2(\theta)}} d(\theta) \\ & \frac{1}{16} \int \frac{\tan(\theta)}{\sec(\theta)\tan(\theta)} d(\theta) \\ & \frac{1}{16} \int \cos(\theta) d\theta \\ & \frac{1}{16} \sin(\theta) + C \end{aligned}$$

Recall that we substituted $x = 4 \sec(\theta)$, which we can rewrite as $\sec(\theta) = \frac{x}{4} = \frac{\text{hyp}}{\text{adj}}$. If we imagine a triangle in which the hypotenuse is x and the adjacent side is 4, then the opposite side must be $\sqrt{16 - x^2}$.



Therefore: $\sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{16-x^2}}{4}$

$$\begin{aligned} \frac{1}{16} \sin(\theta) + C &= \frac{1}{16} \frac{\sqrt{16 - x^2}}{4} + C \\ &= \frac{\sqrt{16 - x^2}}{64} + C \end{aligned}$$

Practice Problem 15

$$\int x^2 \sqrt{a^2 - x^2} dx$$

$$\text{Let : } x = a \sin(\theta) d\theta$$

$$dx = a \cos(\theta) d\theta$$

$$\int a^2 \sin^2(\theta) \sqrt{a^2 - a^2 \sin^2(\theta)} a \cos(\theta) d\theta$$

$$a^4 \int \sin^2(\theta) \cos(\theta) \sqrt{1 - \sin^2(\theta)} d\theta$$

$$a^4 \int \sin^2(\theta) \cos(\theta) \sqrt{\cos^2(\theta)} d\theta$$

$$a^4 \int \sin^2(\theta) \cos^2(\theta) d\theta$$

$$a^4 \int (\sin(\theta) \cos(\theta))^2 d\theta$$

Using the double angle formulas:

$$a^4 \int \left(\frac{\sin(2\theta)}{2}\right)^2 d\theta$$

$$\frac{a^4}{4} \int \sin^2(2\theta) d\theta$$

Using the double angle formulas again:

$$\frac{a^4}{4} \int \frac{1 - \cos(4\theta)}{2} d\theta$$

$$\frac{a^4}{8} \int 1 - \cos(4\theta) d\theta$$

$$\frac{a^4}{8} \left[\theta - \frac{1}{4} \sin(4\theta) \right] + C$$

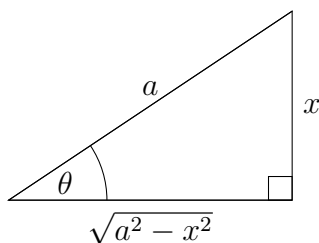
$$\frac{a^4}{8} \left[\theta - \frac{1}{4} 2 \sin(2\theta) \cos(2\theta) \right] + C$$

$$\frac{a^4}{8} \left[\theta - \frac{1}{2} (2 \sin(\theta) \cos(\theta)) (1 - 2 \sin^2(\theta)) \right] + C$$

$$\frac{a^4}{8} \left[\theta - \sin(\theta) \cos(\theta) - 2 \sin^2(\theta) \sin(\theta) \cos(\theta) \right] + C$$

$$\frac{a^4}{8} \left[\theta - \sin(\theta) \cos(\theta) - 2 \sin^3(\theta) \cos(\theta) \right] + C$$

Recall that we substituted $x = a \sin(\theta)$, which we can rewrite as $\sin(\theta) = \frac{x}{a} = \frac{\text{opp}}{\text{hyp}}$. If we imagine a triangle in which the opposite side is x and the hypotenuse is a , then the adjacent side must be $\sqrt{a^2 - x^2}$.



Given this information:

$$\begin{aligned}\theta &= \sin^{-1}\left(\frac{x}{a}\right) \\ \sin(\theta) &= \frac{x}{a} \\ \cos(\theta) &= \frac{\sqrt{a^2 - x^2}}{a}\end{aligned}$$

We can substitute this back into our solution:

$$\begin{aligned}&\frac{a^4}{8} \left[\theta - \sin(\theta) \cos(\theta) - 2 \sin^3(\theta) \cos(\theta) \right] + C \\ &\frac{a^4}{8} \left[\sin^{-1}\left(\frac{x}{a}\right) - \frac{x\sqrt{a^2 - x^2}}{a^2} - \frac{2x^3\sqrt{a^2 - x^2}}{a^4} \right] + C\end{aligned}$$

Practice Problem 21

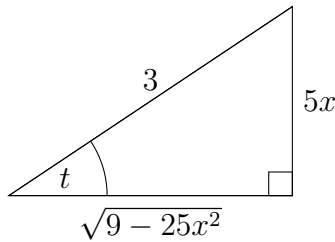
$$\int \frac{x^2}{\sqrt{9 - 25x^2}} dx$$

The terms inside the radical are not of the same form as the problems before. We can rewrite this problem to figure out the substitution.

$$\begin{aligned}&\int \frac{x^2}{\sqrt{3^2 - (5x)^2}} dx \\ &\text{Let : } 5x = 3 \sin(t) \\ &5dx = 3 \cos(t) dt \\ &dx = \frac{3}{5} \cos(t) dt \\ &\int \frac{\left(\frac{3}{5} \sin(t)\right)^2}{\sqrt{3^2 - 3^2 \sin^2(t)}} \frac{3}{5} \cos(t) dt \\ &\frac{27}{125} \int \frac{\cos(t) \sin^2(t)}{\cos(t)} dt \\ &\frac{9}{125} \int \sin^2(t) dt\end{aligned}$$

$$\begin{aligned} & \frac{9}{125} \int \frac{1}{2}(1 - \cos(2t))dt \\ & \frac{9}{250} \left[t - \frac{\sin(2t)}{2} \right] + C \\ & \frac{9}{250} \left[t - \frac{2 \sin(t) \cos(t)}{2} \right] + C \\ & \frac{9}{250} \left[t - \sin(t) \cos(t) \right] + C \end{aligned}$$

Recall that we substituted $5x = 3 \sin(t)$, which we can rewrite as $\sin(t) = \frac{5x}{3} = \frac{\text{opp}}{\text{hyp}}$. If we imagine a triangle in which the opposite side is $5x$ and the hypotenuse is 3 , then the adjacent side is $\sqrt{9 - 25x^2}$.



Therefore: $\cos(t) = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{9-25x^2}}{3}$ and $t = \sin^{-1}\left(\frac{5x}{3}\right)$

$$\begin{aligned} & \frac{9}{250} \left[t - \sin(t) \cos(t) \right] + C \\ & \frac{9}{250} \left[\sin^{-1}\left(\frac{5x}{3}\right) - \frac{5x}{3} \frac{\sqrt{9 - 25x^2}}{3} \right] + C \\ & \frac{9}{250} \left[\sin^{-1}\left(\frac{5x}{3}\right) - \frac{5x\sqrt{9 - 25x^2}}{9} \right] + C \end{aligned}$$

Practice Problem 27

$$\int \sqrt{x^2 + 2x} dx$$

This problem requires a different approach. We need to turn this into the form of a trigonometric substitution problem.

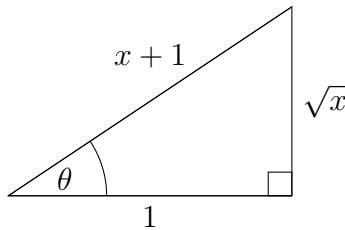
$$\int \sqrt{x^2 + 2x} dx = \int \sqrt{x^2 + 2x + (1 - 1)} dx$$

$$\begin{aligned}
& \int \sqrt{x^2 + 2x + 1} - 1 dx \\
& \int \sqrt{(x + 1)^2 - 1} dx \\
& \text{Let : } x + 1 = \sec(\theta) \\
& dx = \sec(\theta) \tan(\theta) d\theta \\
& \int \sqrt{\sec^2(\theta) - 1} \sec(\theta) \tan(\theta) d\theta \\
& \int \sqrt{\tan^2(\theta)} \sec(\theta) \tan(\theta) d\theta \\
& \int \tan^2(\theta) \sec(\theta) d\theta
\end{aligned}$$

Now we use integration by parts:

$$\begin{aligned}
& \text{Let : } f(x) = \tan(\theta) \quad g'(x) = \tan(\theta) \sec(\theta) d\theta \\
& f'(x) = \sec^2(\theta) d\theta \quad g(x) = \sec(\theta) \\
& \int \tan^2(\theta) \sec(\theta) d\theta = \tan(\theta) \sec(\theta) - \int \sec^2(\theta) \sec(\theta) d\theta \\
& \int \tan^2(\theta) \sec(\theta) d\theta = \tan(\theta) \sec(\theta) - \int (\tan^2(\theta) + 1) \sec(\theta) d\theta \\
& \int \tan^2(\theta) \sec(\theta) d\theta = \tan(\theta) \sec(\theta) - \int \tan^2(\theta) \sec(\theta) d\theta + \int \sec(\theta) d\theta \\
& 2 \int \tan^2(\theta) \sec(\theta) d\theta = \tan(\theta) \sec(\theta) - \int \sec(\theta) d\theta \\
& 2 \int \tan^2(\theta) \sec(\theta) d\theta = \tan(\theta) \sec(\theta) - \ln |\sec(x) + \tan(x)| + C \\
& \int \tan^2(\theta) \sec(\theta) d\theta = \frac{1}{2} \tan(\theta) \sec(\theta) - \frac{1}{2} \ln |\sec(x) + \tan(x)| + C
\end{aligned}$$

Recall that we substituted $x + 1 = \sec(\theta)$, which we can rewrite as $\sec(\theta) = \frac{x+1}{1} = \frac{\text{hyp}}{\text{adj}}$. If we imagine a triangle in which the hypotenuse is $x + 1$ and the adjacent side is 1, then the opposite side must be \sqrt{x} .



Therefore: $\tan(\theta) = \frac{opp}{adj} = \frac{\sqrt{x}}{1}$

$$\begin{aligned}\frac{1}{2} \tan(\theta) \sec(\theta) - \frac{1}{2} \ln |\sec(x) + \tan(x)| + C &= \frac{1}{2} \sqrt{x}(x+1) - \frac{1}{2} \ln |(x+1) + \sqrt{x}| + C \\ &= \frac{\sqrt{x}(x+1)}{2} - \frac{\ln |x + \sqrt{x} + 1|}{2} + C\end{aligned}$$

Practice Problem 45

$$\int x^3 \sqrt{1+x^2} dx$$

For this problem, we can solve it with regular substitution:

$$\begin{aligned}\int x^2 x \sqrt{1+x^2} dx \\ \text{Let : } 1+x^2 &= t \\ x dx &= \frac{dt}{2} \\ \frac{1}{2} \int (t-1)t^{\frac{1}{2}} dt \\ \frac{1}{2} \int t^{\frac{3}{2}} - t^{\frac{1}{2}} dt \\ &= \frac{1}{2} \left[\frac{t^{\frac{5}{2}}}{\frac{5}{2}} - \frac{t^{\frac{3}{2}}}{\frac{2}{2}} + C \right]\end{aligned}$$

But we can also use trigonometric substitution:

$$\begin{aligned}\int x^3 \sqrt{1+x^2} dx \\ \text{Let : } x &= \tan(\theta) \\ dx &= \sec^2(\theta) d\theta \\ \int \tan^3(\theta) \sqrt{1+\tan^2(\theta)} \sec^2(\theta) d\theta \\ \int \tan^3(\theta) \sqrt{\sec^2(\theta)} \sec^2(\theta) d\theta\end{aligned}$$

$$\begin{aligned}
& \int \tan^3(\theta) \sec^3(\theta) d\theta \\
& \int \tan(\theta) \tan^2(\theta) \sec^3(\theta) d\theta \\
& \int \tan(\theta) (\sec^2 - 1) \sec^3(\theta) d\theta \\
& \int \tan(\theta) \sec^5(\theta) d\theta - \int \tan(\theta) \sec^3(\theta) d\theta \\
& \int \tan(\theta) \sec(\theta) \sec^4(\theta) d\theta - \int \tan(\theta) \sec(\theta) \sec^2(\theta) d\theta
\end{aligned}$$

$$\text{Let : } u = \sec(\theta)$$

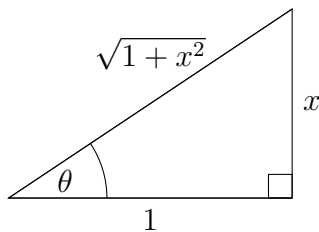
$$du = \sec(\theta) \tan(\theta) d\theta$$

$$\int u^4 du - \int u^2 du$$

$$\frac{u^5}{5} - \frac{u^3}{3} + C$$

$$\frac{\sec^5(\theta)}{5} - \frac{\sec^3(\theta)}{3} + C$$

Recall that we substituted $x = \tan(\theta)$, which we can rewrite as $\tan(\theta) = \frac{x}{1} = \frac{\text{opp}}{\text{adj}}$. If we imagine a triangle in which the opposite side is x and the adjacent side is 1, then the hypotenuse must be $\sqrt{1+x^2}$.



Therefore: $\sec(\theta) = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{1+x^2}}{1}$.

$$\begin{aligned}
\frac{\sec^5(\theta)}{5} - \frac{\sec^3(\theta)}{3} + C &= \frac{1}{5}(\sqrt{1+x^2})^5 - \frac{1}{3}(\sqrt{1+x^2})^3 + C \\
&= \frac{(1+x^2)^{\frac{5}{2}}}{5} - \frac{(1+x^2)^{\frac{3}{2}}}{3} + C
\end{aligned}$$

Follow-up Questions

$$\int x^2 \sqrt{3 - 2x - x^2} dx$$

$$\int x^2 \sqrt{-(x^2 + 2x - 3)} dx$$

$$\int x^2 \sqrt{-(x^2 + 2x - 3 + 4 - 4)} dx$$

$$\int x^2 \sqrt{-(x^2 + 2x + 1) + 4} dx$$

$$\int x^2 \sqrt{4 - (x + 1)^2} dx$$

$$\text{Let : } x + 1 = 2 \cos(\theta)$$

$$dx = -2 \sin(\theta) d\theta$$

$$\int (2 \cos(\theta) - 1)^2 \sqrt{4 - (2 \cos(\theta))^2} (-2 \sin(\theta)) d\theta$$

$$-2 \int (2 \cos(\theta) - 1)^2 \sqrt{4} \sqrt{1 - \cos^2(\theta)} (-2 \sin(\theta)) d\theta$$

$$-4 \int (2 \cos(\theta) - 1)^2 \sqrt{\sin^2(\theta)} (-\sin(\theta)) d\theta$$

$$-4 \int (4 \cos^2(\theta) - 4 \cos(\theta) + 1) (-\sin^2(\theta)) d\theta$$

$$4 \int 4 \cos^2(\theta) \sin^2(\theta) - 4 \cos(\theta) \sin^2(\theta) + \sin^2(\theta) d\theta$$

$$16 \int \cos^2(\theta) \sin^2(\theta) d\theta - 16 \int \cos(\theta) \sin^2(\theta) d\theta - 4 \int \sin^2(\theta) d\theta$$

$$4 \int (2 \cos^2(\theta) \sin^2(\theta))^2 d\theta - 16 \int \cos(\theta) \sin^2(\theta) d\theta - 2 \int 2 \sin^2(\theta) d\theta$$

$$4 \int \sin^2(2\theta) d\theta - 16 \int \cos(\theta) \sin^2(\theta) d\theta - 2 \int 1 - \cos(2\theta) d\theta$$

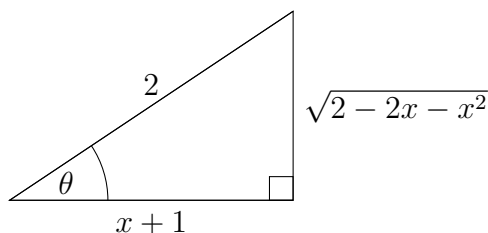
$$\left[2 \int 2 \sin^2(2\theta) d\theta \right] - \left[16 \frac{\cos^3(\theta)}{3} \right] - \left[(2\theta - \sin(2\theta)) \right] + C$$

$$\left[\int 1 - \cos(4\theta) d\theta \right] - \left[16 \frac{\cos^3(\theta)}{3} \right] - \left[(2\theta - \sin(2\theta)) \right] + C$$

$$\left[2\theta - \frac{\sin(4\theta)}{2} \right] - \left[16 \frac{\cos^3(\theta)}{3} \right] - \left[(2\theta - \sin(2\theta)) \right] + C$$

$$\begin{aligned}
& 2\theta - \frac{\sin(4\theta)}{2} - \frac{16 \cos^3(\theta)}{3} - 2\theta + \sin(2\theta) + C \\
& \sin(2\theta) - \frac{\sin(4\theta)}{2} - \frac{16 \cos^3(\theta)}{3} + C \\
& 2 \sin(\theta) \cos(\theta) - \frac{2 \sin(2\theta) \cos(2\theta)}{2} - \frac{16 \cos^3(\theta)}{3} + C \\
& 2 \sin(\theta) \cos(\theta) - \frac{2(2 \sin(\theta) \cos(\theta))(2 \cos^2(\theta) - 1)}{2} - \frac{16 \cos^3(\theta)}{3} + C \\
& 2 \sin(\theta) \cos(\theta) - (2 \sin(\theta) \cos(\theta))(2 \cos^2(\theta) - 1) - \frac{16 \cos^3(\theta)}{3} + C
\end{aligned}$$

Recall that we substituted $x + 1 = 2 \cos(\theta)$, which we can rewrite as $\cos(\theta) = \frac{x+1}{2} = \frac{\text{adj}}{\text{hyp}}$. If we imagine a triangle in which the adjacent side is $x + 1$ and the hypotenuse is 2, then the opposite side must be $\sqrt{4 - (x + 1)^2}$, or $\sqrt{2 - 2x - x^2}$.



Therefore: $\sin(\theta) = \frac{\sqrt{2-2x-x^2}}{2}$

$$\begin{aligned}
& 2 \sin(\theta) \cos(\theta) - (2 \sin(\theta) \cos(\theta))(2 \cos^2(\theta) - 1) - \frac{16 \cos^3(\theta)}{3} + C \\
& 2 \frac{\sqrt{2 - 2x - x^2}}{2} \frac{x + 1}{2} - (2 \frac{\sqrt{2 - 2x - x^2}}{2} \frac{x + 1}{2}) (2 (\frac{x + 1}{2})^2 - 1) - \frac{16 (\frac{x+1}{2})^3}{3} + C \\
& \frac{(x + 1) \sqrt{2 - 2x - x^2}}{2} - \frac{(x + 1) \sqrt{2 - 2x - x^2}}{2} (\frac{(x + 1)^2}{2} - 1) - \frac{16 (\frac{x + 1}{2})^3}{3} + C \\
& = \frac{-(x + 1)^3 \sqrt{2 - 2x - x^2}}{2} - \frac{2(x + 1)^3}{3} + C
\end{aligned}$$

If any errors are found, please contact me at alvin.lin.dev@gmail.com