

Trigonometric Integrals

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Trigonometric Integrals

Using concepts from Trigonometric Substitution, we can solve integrals with trigonometric functions by separating out even powers and using the following identities:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

If we can isolate an even power of sin, we can substitute:

$$\sin^2(\theta) = 1 - \cos^2(\theta)$$

If we can isolate an even power of cos, we can substitute:

$$\cos^2(\theta) = 1 - \sin^2(\theta)$$

The same rules apply for even powers of tan, sec, cot, and csc.

Practice Problem 2

$$\int \sin^3(\theta) \cos^4(\theta) d\theta$$

$$\int \sin^2(\theta) \sin(\theta) \cos^3(\theta) d\theta$$

$$\text{Let } : x = \cos(\theta)$$

$$\begin{aligned}
& dx = -\sin(\theta)d\theta \\
& - \int (1 - \cos^2(\theta)) \cos^4(\theta) dx \\
& - \int (1 - x^2)x^4 dx \\
& - \int x^4 - x^6 dx \\
& -\frac{1}{5}x^5 + \frac{1}{7}x^7 + C \\
& = -\frac{1}{5} \cos^5(\theta) + \frac{1}{7} \cos^7(\theta) + C
\end{aligned}$$

Practice Problem 4

$$\begin{aligned}
& \int_0^{\pi/2} \sin^5(x) dx \\
& \int_0^{\pi/2} \sin^4(x) \sin(x) dx \\
& \int_0^{\pi/2} (1 - \cos^2(x))^2 \sin(x) dx \\
& \int_0^{\pi/2} (\cos^4(x) - 2\cos^2(x) + 1) \sin(x) dx
\end{aligned}$$

$$\text{Let } t = \cos(x)$$

$$dt = -\sin(x) dx$$

The limits from 0 to $\pi/2$ become 1 to 0. $\left(\int_0^{\pi/2} \rightarrow \int_1^0 \right)$

$$- \int_1^0 t^4 - 2t^2 + 1 dt$$

We can flip the integral and negate it to remove the minus sign.

$$\begin{aligned}
& \int_0^1 t^4 - 2t^2 + 1 dt \\
& \left[\frac{1}{5}t^5 - \frac{2}{3}t^3 + t \right]_0^1 \\
& \frac{1}{5} - \frac{2}{3} + 1 = \frac{3 - 10 + 15}{15} \\
& = \frac{8}{15}
\end{aligned}$$

Practice Problem 13

$$\begin{aligned} & \int \sqrt{\cos(\theta)} \sin^3(\theta) d\theta \\ & \int \sqrt{\cos(\theta)} (1 - \sin^2(\theta)) \sin(\theta) d\theta \\ & \text{Let : } t = \cos(\theta) \\ & dt = -\sin(\theta) d\theta \\ & - \int t^{\frac{1}{2}} (1 - t^2) dt \\ & \int -t^{\frac{1}{2}} + t^{\frac{5}{2}} dt \\ & -\frac{2}{3} t^{\frac{3}{2}} + \frac{2}{7} t^{\frac{7}{2}} + C \\ & -\frac{2}{3} \cos^{\frac{3}{2}}(\theta) + \frac{2}{7} \cos^{\frac{7}{2}}(\theta) + C \end{aligned}$$

Practice Problem 21

$$\begin{aligned} & \int \tan(x) \sec^3(x) dx \\ & \int \tan(x) \sec(x) \sec^2(x) dx \\ & \text{Let : } t = \sec(x) \\ & dt = \sec(x) \tan(x) dx \\ & \int t^2 dt \\ & \frac{1}{3} t^3 + C \\ & = \frac{1}{3} \sec^3(x) + C \end{aligned}$$

Practice Problem 22

$$\int \tan^2(\theta) \sec^4(\theta) d\theta$$

$$\int \tan^2(\theta) \sec^2(\theta) \sec^2(\theta) d\theta$$

$$\int \tan^2(\theta)(1 + \tan^2(\theta)) \sec^2(\theta) d\theta$$

$$\text{Let } t = \tan(\theta)$$

$$dt = \sec^2(\theta)$$

$$\int t^2(1 + t^2) dt$$

$$\int t^2 + t^4 dt$$

$$\frac{1}{3}t^3 + \frac{1}{5}t^5 + C$$

$$\frac{1}{3}\tan^3(\theta) + \frac{1}{5}\tan^5(\theta) + C$$

If any errors are found, please contact me at alvin.lin.dev@gmail.com