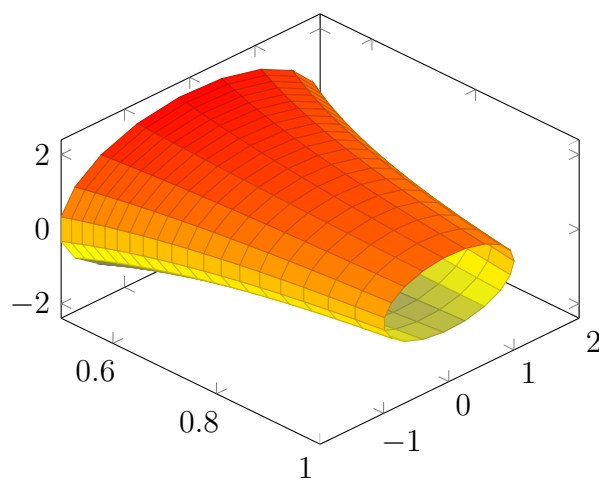


Volumes By Integration (Disks)

Alvin Lin

Calculus II: August 2016 - December 2016

Volumes By Integration (Disks)

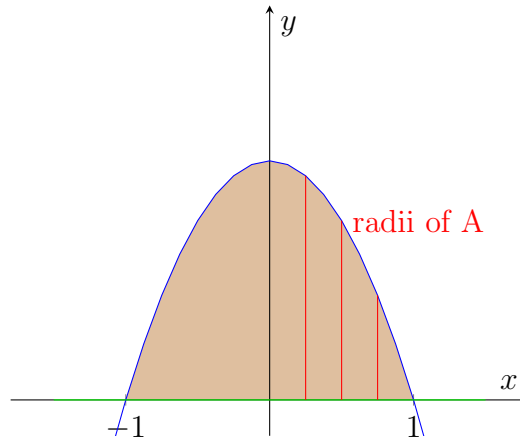


We can take the volume of the above solid by taking areas of cross-sectional slices (A_i) of the volume.

$$\begin{aligned}V_i &= A_i \Delta x \\V &\approx \sum_{i=1}^n A_i \Delta x \approx \lim_{i=1}^n V_i \\V &= \lim_{n \rightarrow \infty} \sum_{i=1}^n V_i = \int_a^b A dx\end{aligned}$$

Example 1

Find the volume of $y = 1 - x^2$ rotated about the x-axis.



$$V = \int A dx$$

$$V = \int \pi r^2 dx$$

$$V = \int_{-1}^1 \pi(1 - x^2)^2 dx$$

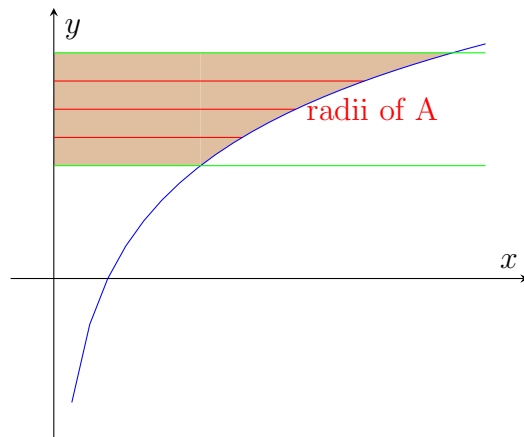
$$V = \pi \int_{-1}^1 1 - x^4 - 2x^2 dx$$

$$V = \pi \left[x + \frac{x^5}{5} - \frac{2x^3}{3} \right]_{-1}^1$$

$$V = \frac{16\pi}{15}$$

Example 2

Find the volume of $y = \ln(x)$ bounded by the lines $y = 1$, $y = 2$, $x = 0$ when rotated about the y-axis.



$$V = \int A dy$$

$$V = \int \pi r^2 dy$$

$$V = \int_1^2 \pi (x)^2 dy$$

$$V = \int_1^2 \pi (e^y)^2 dy$$

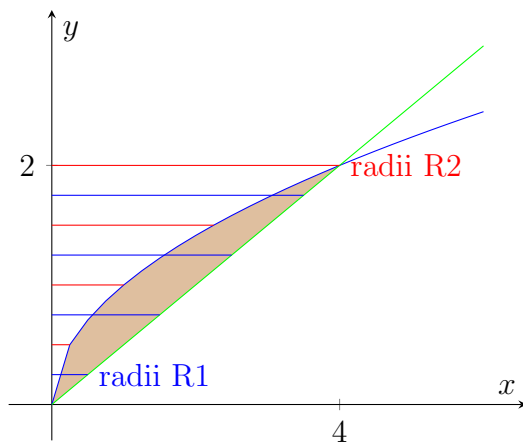
$$V = \pi \int_1^2 e^{2y} dy$$

$$V = \pi \left[\frac{e^{2y}}{2} \right]_1^2$$

$$V = \frac{\pi}{2} [e^4 - e^2]$$

Example 3

Find the volume of the region enclosed between $y^2 = x$ and $x = 2y$ when it is rotated about the y-axis.



$$V = \int A dy$$

$$V = \int \pi(R1)^2 - (R2)^2 dy$$

$$V = \pi \int_0^2 (2y)^2 - (y^2)^2 dy$$

$$V = \pi \int_0^2 4y^2 - y^4 dy$$

$$V = \pi \left[\frac{4y^3}{3} - \frac{y^5}{5} \right]_0^2$$

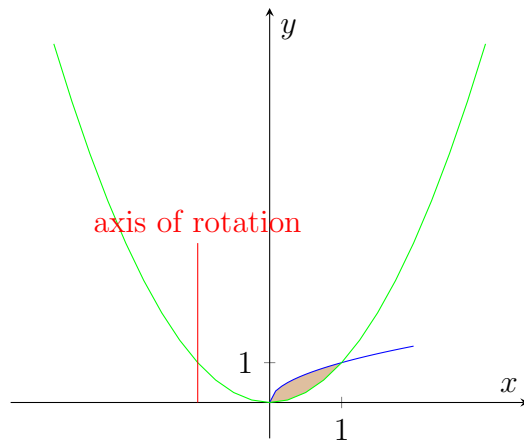
$$V = \frac{32\pi}{3} - \frac{32\pi}{5}$$

Example 4

$$y = x^2 \quad x = y^2$$

rotated about:

$$x = -1$$



$$V = \int A dy$$

$$V = \int \pi(R1)^2 - (R2)^2 dy$$

$$V = \pi \int_0^1 [(1+x)^2 - (1+x)^2] dy$$

$$V = \pi \int_0^1 [(1+\sqrt{y})^2 - (1+y^2)^2] dy$$

$$V = \pi \int_0^1 [1+y+2\sqrt{y} - 1 - y^4 - 2y^2] dy$$

$$V = \pi \left[y + \frac{y^2}{2} + \frac{4y^{3/2}}{3} - y - \frac{y^5}{5} - \frac{2y^3}{3} \right]$$

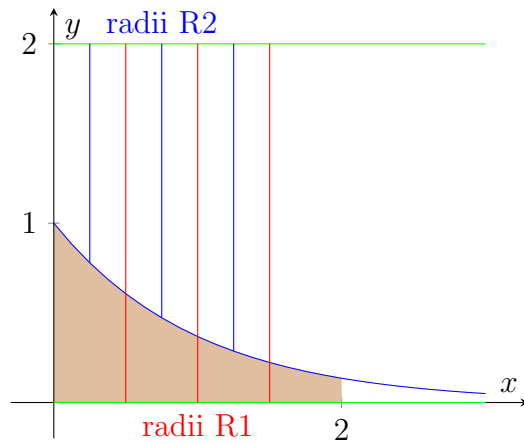
$$V = \frac{29\pi}{30}$$

Practice Problem 12

$$y = e^{-x} \quad y = 1 \quad x = 2$$

rotated about:

$$y = 2$$



$$V = \int A dx$$

$$V = \int \pi(R1)^2 - (R2)^2 dy$$

$$V = \pi \int_0^2 [(2 - y)^2 - 1^2] dx$$

$$V = \pi \int_0^2 [(2 - e^{-x})^2 - 1] dx$$

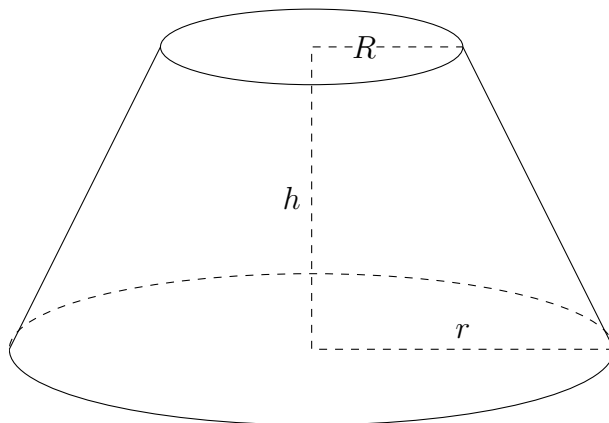
$$V = \pi \int_0^2 [4 - e^{-2x} - 4e^{-x} - 1] dx$$

$$V = \pi \int_0^2 [3 + e^{2x} - 4e^{-x}] dx$$

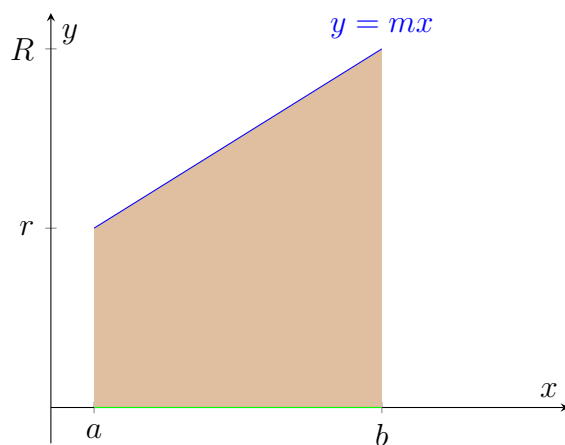
$$V = \pi \left[3x + \frac{e^{-2x}}{-2} + 4e^{-x} \right]_0^2$$

$$V = \pi \left[\left(6 - \frac{1}{2}e^{-4} + 4e^{-2} \right) - \left(\frac{-1}{2} + 4 \right) \right]$$

Volume of a Frustum



We can turn this problem into an integration problem by taking a slice of the frustum and revolving it around the x-axis.



$$a - b = h \quad y = mx$$

$$r = ma \quad R = mb$$

$$V = \int A dx$$

$$V = \int \pi r^2 dx$$

$$V = \pi \int_a^b (y)^2 dx$$

$$V = \pi \int_a^b (mx)^2 dx$$

$$V = \pi(m^2) \left[\frac{x^3}{3} \right]_a^b$$

$$V = \pi(m^2) \left(\frac{b^3}{3} - \frac{a^3}{3} \right)$$

$$V = \frac{\pi(m^2)}{3} (b^3 - a^3)$$

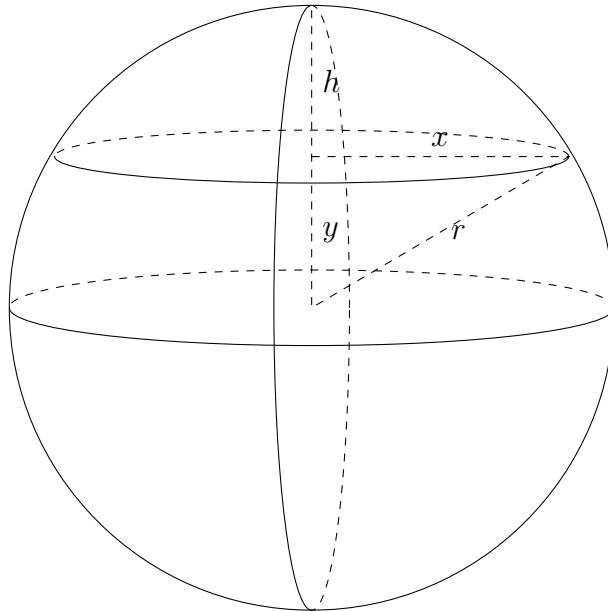
$$V = \frac{\pi(m^2)}{3} (b - a)(b^2 + ab + a^2)$$

Since we know $b - a = h$, $r = ma$, and $R = mb$:

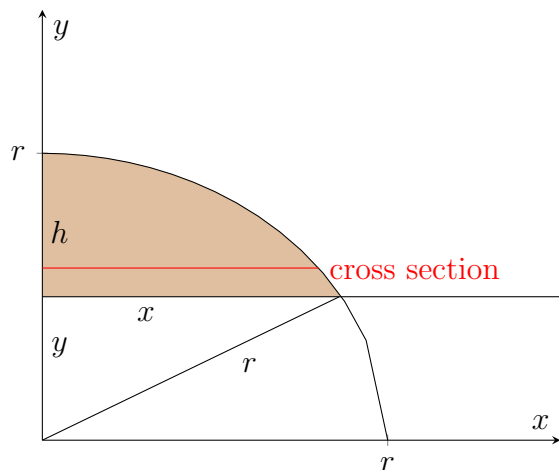
$$V = \frac{\pi(m^2)h}{3} \left(\left(\frac{r}{m} \right)^2 + \frac{Rr}{m^2} + \left(\frac{R}{m} \right)^2 \right)$$

$$V = \frac{\pi h}{3} [r^2 + R^2 + Rr]$$

Volume of a Section of a Sphere



The same proof as above can be applied to most shapes. For example, a section of a sphere:



From the Pythagorean theorem, we know that $x^2 + y^2 = r^2$, thus it follows that $x^2 = r^2 - y^2$.

$$V = \int A dy$$

$$V = \int \pi x^2 dy$$

$$V = \int_{r-h}^r \pi(r^2 - y^2) dy$$

$$V = \pi \left[r^2 y - \frac{y^3}{3} \right]_{r-h}^r$$

$$V = \pi \left(h^2 r - \frac{h^3}{3} \right)$$

$$V = \pi h^2 \left(r - \frac{1}{3} h \right)$$

If any errors are found, please contact me at alvin.lin.dev@gmail.com