Probability and Statistics

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Statistics and their Distributions

The random variables $X_1, X_2, \dots, X_n$ are said to form a random sample of size n if the following conditions are satisfied.

1. Every X_i has the same probability distribution.
2. The random variable $X_1, X_2, \dots, X_n$ are independent.

Example

Using the popuulation of graduating RIT students, select a person from the population and observe his or her GPA. Then, with/without replacement (the population size is big and n is not big compared to the population size) select a student and then observe his or her GPA. Repeat n times. We consider n random variables where X_1 is the GPA of the first selected person, X_2 is the GPA of the second selected person, and so on. Then $X_1, X_2, \dots, X_n$ for a random sample of size n. Once we finish the n^{th} observation, we have n numbers:

$$x_1, x_2, \dots, x_n$$

We can consider the sample mean, median, variance, fourth spread, etc as functions. Consider the case $n = 3$.

$$(2.9, 3.0, 3.8) \rightarrow 3.233\dots$$
$$(2.5, 3.7, 3.9) \rightarrow 3.466\dots$$

$\overline{X}$ is a statistic and random variable that takes a sample as input and outputs a number. Sample mean, sample median, sample variance, et cetera can be viewed as random variables. Each of them is called a statistic. The probability distribution of $X_1, X_2, \dots, X_n$ is also called population distribution.

Example

A certain brand of MP3 player comes in three configurations: a model with 2GB of memory, costing \$80, a 4GB model priced at \$100, and an 8GB version with a price tag of \$120. If 20% of all purchasers choose the 2GB model, 30% choose the 4GB model, then the probability distribution of the cost X of a single randomly selected MP3 player purchase is given by:

x	80	100	120
$p(x)$	0.2	0.3	0.5

$$
\begin{aligned}
p(80) &= p(X = 80) = 0.2 \\
\mu &= 106 \\
\sigma^2 &= 244
\end{aligned}
$$

Suppose on a particular day, only two MP3 players are sold. Let X_1 be the random variable representing the revenue of the first sale, and X_2 be the random variable representing the revenue of the second sale. X_1 and X_2 are independent and their pmf is $p(x)$. X_1 and X_2 form a random sample of size 2.

x_1	x_2	$p(x_1, x_2)$	$\overline{x}$	s^2
80	80	0.04	80	0
80	100	0.06	90	200
80	120	0.2×0.5	100	800
100	80	0.06	90	200

The Distribution of the Sample Mean

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and standard deviation σ. Then:

1. $E(\overline{X}) = \mu_{\overline{x}} = \mu$
2. $V(\overline{X}) = \sigma^2_{\overline{x}} = \frac{\sigma^2}{n}$

Let $T_\circ = X_1 + X_2 + X_3 + \cdots + X_n$. (the sample total)

1. $E(T_\circ) = \mu_{T_\circ} = n\mu$
2. $V(T_\circ) = \sigma^2_{T_\circ} = n\sigma^2$

Each of $X_1, X_2, \ldots, X_n$ has pmf $p(x)$ or pdf $f(x)$. $X_1, X_2, X_3, \ldots, X_n$ are independent.

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

In the above proposition, $p(x)$ or $f(x)$ can be any function satisfying the following conditions.

- $p(x) \geq 0$ for any x.
- $f(x) \geq 0$ for any x.
- $\sum p(x) = 1$
- $\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = 1$

$p(x)$ or $f(x)$ is the pmf or pdf of X_i and represents the population distribution. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2. Then for any n, $\overline{X}$ is normally distributed with mean $\mu_{\overline{x}} = \mu$ and variance $\sigma^2_{\overline{x}} = \frac{\sigma^2}{n}$ (same as before). $T_\circ$ is normally distributed with $\mu_{T_\circ} = n\mu$ and $\sigma^2_{T_\circ} = n\sigma^2$.

Central Limit Theorem

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Then if n is sufficiently large, $\overline{X}$ has approximately normal distribution with $\mu_{\overline{x}} = \mu$ and $\sigma^2_{\overline{x}} = \frac{\sigma^2}{n}$. $T_\circ$: normal with $\mu_{T_\circ} = n\mu$ and $\sigma^2_{T_\circ} = n\sigma^2$.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech