

Probability and Statistics

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Standard Normal Distribution

Standard Normal Distribution is the normal distribution within $\mu = 0$ and $\sigma = 1$. The pdf of a standard normal random variable Z is:

$$f(\delta; 0, \sigma) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\delta^2}{2}}$$

The cdf of Z is:

$$\begin{aligned}\Phi(\delta) &= P(Z \leq \delta) \\ &= \int_{-\infty}^{\delta} f(y; 0, 1) \, dy \\ &= \int_{-\infty}^{\delta} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy\end{aligned}$$

Example

Find $P(Z \leq 1.25)$, where Z is the standard normal random variable.

$$\begin{aligned}P(Z \leq 1.25) &= \int_{-\infty}^{1.25} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy \\ &= \Phi(1.25) \\ &\approx 0.8944\end{aligned}$$

Example

Find $\eta(0.99)$, 99th percentile, for the standard normal variable Z .

$$\begin{aligned}0.99 &= \int_{-\infty}^{\eta(0.99)} f(\delta; 0, 1) \, d\delta \\ &= \Phi(\eta(0.99)) \\ \eta(0.99) &= 2.33\end{aligned}$$

The Z_α

Z_α denotes the value on the z axis for which the α of the area under the z curve lies to the right of Z_α . Example:

$$\begin{aligned}Z_{0.1} &= \eta(0.9) \\ 0.9 &= \Phi(Z_{0.1}) \\ &= \int_{-\infty}^{\eta(0.9)} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy \\ Z_{0.1} &= 1.28\end{aligned}$$

Relationship between normal distribution and standard normal distribution

Let X be the normal random variable pdf $f(x; \mu, \sigma)$, and Z be the standard normal random variable (with pdf $f(\delta; 0, 1)$). Derivation:

$$F(x; \mu, \sigma) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

$$\Phi(\delta) = \int_{-\infty}^{\delta} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$Z = \frac{X - \mu}{\sigma}$$

$$\begin{aligned} P(a \leq X \leq b) &= P\left(\frac{a - \mu}{\sigma} \leq Z \leq \frac{b - \mu}{\sigma}\right) \\ &= P\left(Z \leq \frac{b - \mu}{\sigma}\right) - P\left(Z \leq \frac{a - \mu}{\sigma}\right) \\ &= \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right) \end{aligned}$$

$$P(X \leq a) = P\left(Z \leq \frac{a - \mu}{\sigma}\right)$$

$$\begin{aligned} P(X \geq b) &= 1 - P(X \leq b) \\ &= 1 - P\left(Z \leq \frac{b - \mu}{\sigma}\right) \\ &= 1 - \Phi\left(\frac{b - \mu}{\sigma}\right) \end{aligned}$$

Example

The breakdown voltage of a randomly chosen diode of a particular type is known to be normally distributed. Find the probability that a diode's breakdown voltage is within 1 standard deviation of its mean value. Experiment: select a diode and measure its breakdown voltage. X : the measured breakdown voltage.

$$\mu = E(X)$$

$$\sigma = \sqrt{V(X)}$$

$$\begin{aligned}
P(\mu - \sigma \leq X \leq \mu + \sigma) &= \int_{\mu - \sigma}^{\mu + \sigma} f(x; \mu, \sigma) \, dx \\
&= \int_{\mu - \sigma}^{\mu + \sigma} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx \\
&= P\left(\frac{(\mu - \sigma) - \mu}{\sigma} \leq Z \leq \frac{(\mu + \sigma) - \mu}{\sigma}\right) \\
&= P(-1 \leq Z \leq 1) \\
&= \Phi(1) - \Phi(-1) \\
&= 0.8413 - 0.1587
\end{aligned}$$

Relationship between two percentiles

(100p)th percentile for normal (μ, σ) :

$$= \mu + [(100p)\text{th percentile for standard normal}] \times \sigma$$

Example

The temperature reading from a thermocouple placed in a constant-temperature medium is normally distributed with mean μ , the actual temperature of the medium, and standard deviation σ . What would the value of σ have to be to ensure that 95% of all readings are within $.1^\circ$ of μ ?

Random variable X that has a normal distribution with $E(X) = \mu$ and standard deviation $\sqrt{V(X)} = \sigma$.

$$\begin{aligned}
0.95 &= P(\mu - 0.1 \leq X \leq \mu + 0.1) \\
&= P\left(\frac{(\mu - 0.1) - \mu}{\sigma} \leq Z \leq \frac{(\mu + 0.1) - \mu}{\sigma}\right) \\
&= P\left(-\frac{0.1}{\sigma} \leq Z \leq \frac{0.1}{\sigma}\right)
\end{aligned}$$

$$\Phi\left(\frac{0.1}{\sigma}\right) = 0.975$$

$$\frac{0.1}{\sigma} = 1.96$$

$$\sigma = \frac{0.1}{1.96}$$

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech