Probability and Statistics

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Probability and Statistics: January 2017 - May 2017

The Poisson Probability Distribution

Recall binomial random variables and its probability mass function.

$$b(x;n,p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & ,x = 0,1,2,\dots,n \\ 0 & ,otherwise \end{cases}$$

As $n \to \infty$ and $p \to 0$, $np \to \mu > 0$.

$$b(x;n,p) \to p(x;\mu)$$

Poisson Distribution

$$p(x;\mu) = \begin{cases} \frac{\mathrm{e}^{-\mu}\mu^x}{x!} & ,x = 0,1,2,\dots \\ 0 & ,otherwise \end{cases}$$

The Poisson model is a reasonably good approximation of the binomial model when $n \geq 20$ with $p \leq 0.05$ or $n \geq 100$ with $p \leq 0.10$.

binomial	Poisson
$E(X) = np$	$E(X) = \mu$
$np \sim \mu$	
$V(X) = np(1-p)$	$V(X) = \mu$
$np(1-p) \sim np \sim \mu$	

$$
\begin{aligned}
\sum_{x=0}^{\infty} p(x;\mu) &= \sum_{x=0}^{\infty} \frac{\mathrm{e}^{-\mu}\mu^x}{x!} \\
&= \mathrm{e}^{-\mu} \sum_{x=0}^{\infty} \frac{\mu^x}{x!} \\
&= \mathrm{e}^{-\mu}\mathrm{e}^{\mu} \\
&= \mathrm{e}^{-\mu+\mu} \\
&= \mathrm{e}^{0} \\
&= 1
\end{aligned}
$$

Example

Let X denote the number of traps (defects of a certain kind) in a particular type of semiconductor transistor, and suppose it has Poisson distribution with $\mu = 2$. The probability that there are exactly 3 traps is:

$$
\begin{aligned}
P(X = 3) &= p(3;\mu) \\
&= p(3;2) \\
&= \frac{\mathrm{e}^{-2}(2)^3}{3!} \\
&\approx 0.18
\end{aligned}
$$

Poisson Process

1. There exists a parameter $\alpha > 0$ such that for any short time interval of length Δt, the probability that exactly one event occurs is:

$$\alpha\Delta t + o(\Delta t)$$

2. The probability of more than one event occurring during Δt is $o(\Delta t)$ [which, along with assumption 1, implies that the probability of no events during Δt is $1 - \alpha\Delta t - o(\Delta t)$].

3. The number of events occurring during the time interval Δt is independent of the number that occur prior to this time interval.

4. The probability that the "event" occurs k times in a time interval of length t is:

$$P_k(t) = \frac{\mathrm{e}^{-(\alpha t)}(\alpha t)^k}{k!}$$

where α is the average rate of occurrence of the "event".

Little o notation

x^2 is $o(|x|)$ for $x \to 0$.

$$\frac{x^2}{|x|} \to 0 \ as \ x \to 0$$

$x^3 + x^5$ is $o(|x|^2)$ for $x \to 0$.

$$\frac{x^3 + x^5}{|x|^2} \to 0 \ as \ x \to 0$$

$\sqrt{|x|}$ is not $o(|x|)$ for $x \to 0$.

$$\frac{\sqrt{|x|}}{|x|} = \frac{1}{\sqrt{|x|}} \ as \ x \to 0$$

$\frac{1}{\sqrt{x}}$ is not $o(\frac{1}{x})$ for $x \to 0$.

$$\frac{\frac{1}{\sqrt{x}}}{\frac{1}{x}} \not\to 0 \ as \ x \to \infty$$

$\frac{1}{x^2}$ is $o(\frac{1}{x})$ for $x \to \infty$.

$$\frac{\frac{1}{x^2}}{\frac{1}{x}} \to 0 \ as \ x \to \infty$$

Let $R_n(x) = f(x) - T_n(x)$, where $T_n(x)$ is the nth order Maclaurin polynomial for f. Then $R_n(x) = o(|x|^n)$ for $x \to 0$.

Example

Suppose pulses arrive at a counter at an average rate of six per minute so that $\alpha = 6$. Suppose this is a Poisson process. Find the probability that at least one pulse arrives during a time interval of length 0.5 minutes. Let X be a random variable denoting the number of pulses arrived.

$$P(X \geq 1) = 1 - P(X = 0)$$

$$P(X = 0) = \frac{\mathrm{e}^{-\alpha t}(\alpha t)^k}{k!}$$

$$= \frac{\mathrm{e}^{-(6 \cdot 0.5)}(6 \cdot 0.5)^0}{0!}$$

$$= \mathrm{e}^{-3}$$

$$P(X \geq 1) = 1 - \mathrm{e}^{-3}$$

$$\approx 0.950$$

Tell whether the statement is true:

- $$\sum_{k=0}^{\infty} \frac{\mathrm{e}^{-\alpha t}(\alpha t)^k}{k!} = 1$$

  True, since $P(X = 0) + P(X = 1) + P(X = 2) + \ldots$ is exhaustive.

- The probability that 6 pulses arrive during a time interval of length 1.0 minute is equal to:

  $$[\frac{\mathrm{e}^{-(6 \cdot 0.5)}(6 \cdot 0.5)^3}{3!}] + [\frac{\mathrm{e}^{-(6 \cdot 0.5)}(6 \cdot 0.5)^3}{3!}]$$

  False, they are not additive.

- If 4 pulses arrive during 3:00:00-3:00:30, the probability that 3 pulses arrive during 3:00:30-3:01:00 is equal to:

  $$[\frac{\mathrm{e}^{-(6 \cdot 0.5)}(6 \cdot 0.5)^4}{4!}] \cdot [\frac{\mathrm{e}^{-(6 \cdot 0.5)}(6 \cdot 0.5)^3}{3!}]$$

  False, the events are independent.

- The probability that 6 pulses are received in a time interval of length 1.0 minute is equal to:

  $$\frac{\mathrm{e}^{-6 \cdot 1}(6 \cdot 1)^6}{6!}$$

  True

- The probability that 6 pulses are received in a time interval of length 1.0 minute is equal to:

  $$\sum_{k=0}^{6} [\frac{\mathrm{e}^{-6 \cdot 0.5}(6 \cdot 0.5)^k}{k!}] \cdot [\frac{\mathrm{e}^{-6 \cdot 0.5}(6 \cdot 0.5)^{6-k}}{(6-k)!}]$$

Example

In proof testing of circuit boards, the probability that any particular diode will fail is .01. Suppose a circuit board contains 200 diodes.

- How many diodes would you expect to fail, and what is the standard deviation of the number expected to fail?

$$E(X) = np = 200 \times 0.01 = 2$$

$$\sqrt{V(X)} = \sqrt{np(1-p)} = \sqrt{200 \times 0.01 \times (1 - 0.01)} = 1.407$$

- What is the (approximate) probability that at least four diodes will fail on a randomly selected board?

$$\begin{aligned} P(X \geq 4) &= 1 - P(X \leq 3) \\ &= 1 - \sum_{x=0}^{3} b(x; 200, 0.01) \\ &\approx 1 - \sum_{x=0}^{3} p(x; \mu) \\ \mu &= 200 \times 0.01 = 2 \end{aligned}$$

For a random variable X with its pmf being $p(x; \mu)$, it turns out that $E(X) = \mu$ and $V(X) = \mu$.

Example

Let X be a continuous random variable. The probability distribution or probability density function (pdf) of X is a function $f(x)$ such that for any two numbers a and b with $a \leq b$.

$$P(a \leq X \leq b) = \int_a^b f(x)\ \mathrm{d}x$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech