

# Probability and Statistics

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## Discrete Random Variables

A discrete random variable is a random variable whose possible values either constitute a finite set or else can be listed in an infinite sequence in which there is a first element, a second element, and so on (a countably infinite set). A countable set is finite or countably infinite.

- Finite sets:

$$\{1\}$$
$$\{A, B, E\}$$

- Countably infinite sets:

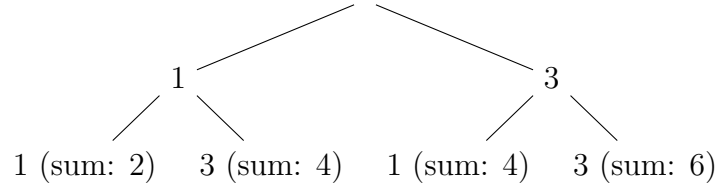
$$\{2, 4, 6, 8, \dots\}$$
$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$$
$$\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$
$$\{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid m \geq 0, n \geq 0\}$$

A random variable is continuous if both of the following apply:

1. Its set of possible values consists either of all members in a single interval on the number line, possibly infinite in extent (eg:  $(-\infty, \infty)$ ,  $(0, \infty)$ ,  $\dots$ ) or all numbers in a disjoint union of such intervals.

## Probability Mass Function

There are two balls marked 1 and 3, respectively. Select a ball twice with replacement.



What is the probability that the sum is 4?

$$\frac{2}{4} = \frac{1}{2}$$

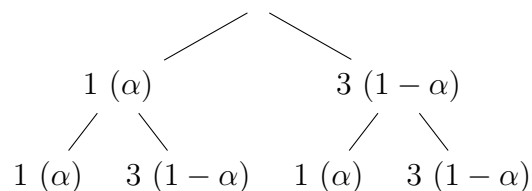
Let  $X$  be the random variable indicating the value of the sum of the two numbers.

$$\begin{aligned} p(4) &= P(X = 4) \\ &= P(\{\omega \in S \mid X(\omega) = 4\}) \\ &= P(\{(1, 3), (3, 1)\}) \\ &= \frac{2}{4} \end{aligned}$$

$p(X = 4)$  is the probability mass function (pmf), a function that gives the probability that a random variable is equal to some exact number.

## Example

Using the same problem as before, the selections are independent and the probability that we get a 1 in a selection is  $\alpha$ .



$$\begin{aligned}
p(2; \alpha) &= P(X = 2) \\
&= \alpha^2 \\
p(4; \alpha) &= P(X = 4) \\
&= \alpha(1 - \alpha) + (1\alpha)\alpha \\
&= 2\alpha(1 - \alpha) \\
p(6; \alpha) &= (1 - \alpha)^2
\end{aligned}$$

$\alpha$  can take various values. The collection of  $p(x; \alpha)$

$$(p(x; \frac{1}{2}), p(x; \frac{1}{6}), \dots)$$

is a family of probability distributions.

## Cumulative Distribution Function

The cumulative distribution function (cdf)  $F(x)$  of a discrete random variable  $X$  with pmf  $p(x)$  is defined for every number  $x$  by:

$$F(x) = P(X \leq x) = \sum_{y; y \leq x} p(y)$$

## Example

Using the same problem as before:

$$F(2) = P(X \leq 2)$$

$$= \frac{1}{4}$$

$$= \sum_{y \leq 2} p(y)$$

$$= p(2)$$

$$= \frac{1}{4}$$

$$F(4) = P(X \leq 4)$$

$$= \sum_{y \leq 4} p(y)$$

$$= p(2) + p(4)$$

$$= \frac{1}{4} + \frac{2}{4}$$

$$= \frac{3}{4}$$

$$F(6) = P(X \leq 6)$$

$$= \sum_{y \leq 6} p(y)$$

$$= p(2) + p(4) + p(6)$$

$$= \frac{1}{4} + \frac{2}{4} + \frac{1}{4}$$

$$= 1$$

$$\begin{aligned}
P(4 \leq X \leq 6) &= \frac{3}{4} \\
&= F(6) - F(2) \\
&= \left(\frac{1}{4} + \frac{2}{4} + \frac{1}{4}\right) - \left(\frac{1}{4}\right) \\
&= \frac{3}{4} \\
P(2 \leq X \leq 4) &= \frac{3}{4} \\
&= F(4) = P(X \leq 4) \\
&= \frac{3}{4}
\end{aligned}$$

For any two numbers  $a$  and  $b$  with  $a \leq b$ :

$$P(a \leq X \leq b) = F(b) - F(a-)$$

where  $a-$  represents the largest possible  $X$  value that is strictly less than  $a$ . Notation:

<i>probability</i>	$P(\text{event} \in S)$
<i>pmf of a random variable</i>	$p(\text{a number})$
<i>cdf of a random variable</i>	$F(\text{a number})$

## Expected Values

Let  $X$  be a discrete random variable with set of possible values  $D$  and pmf  $p(x)$ . The expected value or mean value of  $X$  is:

$$E(X) = \mu_x = \mu = \sum_{x \in D} xp(x)$$

If the random variable  $X$  has a set of possible values  $D$  and pmf  $p(x)$ , then the expected value of any function  $h(X)$  is:

$$E[h(x)] = \sum_{x \in D} h(x)p(x)$$

If  $h(X)$  is of very special type  $h(X) = aX + b$ , where  $a$  and  $b$  are constants:

$$E[h(x)] = E(aX + b) = \mu_{ax+b} = aE(x) + b$$

## Example

Extending from the same problem:

$$\begin{aligned} E(X) &= \mu_x \\ &= \sum_{x \in D} xp(x) \\ &= 2p(2) + 4p(4) + 6p(6) \\ &= 2\frac{1}{4} + 4\frac{2}{4} + 6\frac{1}{4} \\ &= \frac{2 + 8 + 6}{4} \\ &= 4 \end{aligned}$$

$$\begin{aligned} E(e^X) &= \sum_{x \in D} e^x p(x) \\ &= e^2 p(2) + e^4 p(4) + e^6 p(6) \\ &= e^2 \frac{1}{4} + e^4 \frac{2}{4} + e^6 \frac{1}{4} \\ &= \frac{1}{4} [e^2 + 2e^4 + e^6] \end{aligned}$$

$$\begin{aligned} E(10X - 20) &= (10 \times 2 - 20)p(2) + (10 \times 4 - 20)p(4) + (10 \times 6 - 20)p(6) \\ &= 10E(X) - 20 \\ &= 10 \times 4 - 20 \\ &= 20 \end{aligned}$$

## Variance and Standard Deviation

Let  $X$  have pmf  $p(x)$  and expected value  $\mu$ . The variance of  $X$  is:

$$V(X) = \sigma_x^2 = \sigma^2 = \sum_{x \in D} (x - \mu)^2 p(x) = E((x - \mu)^2)$$

The standard deviation of  $X$  is:

$$\sigma_x = \sigma = \sqrt{V(X)} = \sqrt{\sigma_x^2}$$

## Example

Extending from the same problem:

$$\begin{aligned}V(X) &= (2 - 4)^2 p(2) + (4 - 4)^2 p(4) + (6 - 4)^2 p(6) \\&= 2^2 \frac{1}{4} + 0^2 \frac{2}{4} + 2^2 \frac{1}{4} \\&= 2 \\V(10X - 20) &= 10^2 V(X) \\&= 100 \times 2 \\&= 200\end{aligned}$$

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)