

Probability and Statistics

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Conditional Probability

Let A and B be events of an experiment. The conditional probability of A given that B has occurred is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Idea Behind This Formula

Roll a die:

A : *getting a 2*

B : *getting an even number*

$$P(A) = \frac{\# \text{ of outcomes favorable to } A}{\text{total } \# \text{ of outcomes}}$$

(Under the assumption that the outcomes are equally likely)

Let's consider the probability of A given that B has occurred. Suppose the dice have detectors on the faces 2, 4, and 6. After we roll the dice, we can detect whether the dice has landed on a 2, 4, or 6, but we will not know which of the even numbers it has landed on.

$$P(A|B) = \frac{\# \text{ of outcomes favorable to } A \text{ and in } B}{\text{total } \# \text{ of outcomes in } B}$$

$$P(A|B) = \frac{1}{3}$$

Considering the situation in the previous example, find $P(B|A)$.

$$P(B|A) = \frac{1}{1} = 1$$

If $P(B) > 0$, the conditional probability of A given that B has occurred is $P(A|B) = \frac{P(A \cap B)}{P(B)}$. Thus it follows that:

$$P(A \cap B) \times P(B) = P(A|B)$$

This holds true when $P(B) = 0$ as well.

Bayes' Theorem

Let $A_1, A_2, A_3, \dots, A_k$ be events of an experiment. The events $A_1, A_2, A_3, \dots, A_k$ are **mutually exclusive** or **disjoint** if and only if $A_i \cap A_j = \emptyset$ for any $i, j \in \{1, 2, 3, \dots, k\}$ with $i \neq j$. If $A_1, A_2, A_3, \dots, A_k$ are mutually exclusive, then $A_1 \cap A_2 \cap A_3 = \emptyset$, $A_1 \cap A_2 \cap A_4 \cap A_6 = \emptyset$, and so on.

The events $A_1, A_2, A_3, \dots, A_k$ are **exhaustive**, if and only if $A_1 \cup A_2 \cup A_3 \cup \dots \cup A_j = S$. If A_1 and A_2 are exhaustive, then:

$$\begin{aligned} P(A_1) + P(A_2) &= P(A_1 \cup A_2) + P(A_1 \cap A_2) \\ &\text{(Inclusion Exclusion Principle)} \\ &= P(S) + P(A_1 \cap A_2) \\ &\text{(since } A_1 \text{ and } A_2 \text{ are exhaustive)} \\ &= 1 + P(A_1 \cap A_2) \\ &\geq 1 \end{aligned}$$

Let's consider the contrapositive of the statement.

If $P(A_1) + P(A_2) < 1$, then A_1 and A_2 are not exhaustive.

The Law of Total Probability

Let A_1, A_2, \dots, A_k be mutually exclusive and exhaustive events. Then for any other event B :

$$\begin{aligned} P(B) &= P(B|A_1)P(A_1) + \dots + P(B|A_k)P(A_j) \\ &= \sum_{i=1}^k P(B|A_i)P(A_i) \end{aligned}$$

Example

Roll a die:

B : a 2 or a 3

A_1 : a number less than 3

A_2 : a number greater than 2

	B	A_1	A_2	$P(B A_1)$	$P(B A_2)$
1		✓			
2	✓	✓		✓	
3	✓		✓		✓
4			✓		
5			✓		
6			✓		

$$\begin{aligned}P(B) &= \frac{2}{6} \\P(B|A_1)P(A_1) + P(B|A_2)P(A_2) &= \left(\frac{1}{2}\right)\left(\frac{2}{6}\right) + \left(\frac{1}{4}\right)\left(\frac{4}{6}\right) \\&= \frac{1}{6} + \frac{1}{6} \\&= \frac{2}{6}\end{aligned}$$

Consistency Check:

$$A_1 \cap A_2 = \emptyset$$

$$A_1 \cup A_2 = S$$

A_1 and A_2 are mutually exclusive and exhaustive, so the law applies.

Example

An individual has 3 different email accounts. Most of her messages, in fact 70%, come into account #1, whereas 20% come into account #2, and the remaining 10% into account #3. Of the messages into account #1, only 1% are spam, whereas the corresponding percentages for accounts #2 and #3 are 2% and 5%, respectively. What is the probability that a randomly selected message is spam?

$$P(\text{spam}) = (0.7)(0.01) + (0.2)(0.02) + (0.1)(0.05)$$

Is this an application of the Law of Total Probability? Let:

B : an email is spam

A_1 : an email goes to account #1

A_2 : an email goes to account #2

A_3 : an email goes to account #3

$$A_1 \cap A_2 = \emptyset \quad A_1 \cap A_3 = \emptyset \quad A_2 \cap A_3 = \emptyset$$

The events A_1, A_2, A_3 are mutually exclusive.

$$A_1 \cup A_2 \cup A_3 = S$$

The events A_1, A_2, A_3 are exhaustive. So by the Law of Total Probability:

$$\begin{aligned} P(B) &= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3) \\ &= (0.01)(0.7) + (0.02)(0.2) + (0.05)(0.1) \end{aligned}$$

Another visualization

Fill a table with a convenient hypothetical total number of emails.

	account #1	account #2	account #3	total
spam	1×70	2×20	5×10	
not spam	99×70	98×20	95×10	
total	100×70	100×20	100×10	100×100

$$P(B) = \frac{(1)(70) + (2)(20) + (5)(10)}{(100)(100)}$$

We can also compute the conditional probabilities and check the consistency with the Law of Total Probability.

Bayes' Theorem (again)

Let $A_1, A_2, A_3, \dots, A_k$ be a collection of k mutually exclusive and exhaustive events with prior probabilities $P(A_i) (i \in \{1, 2, \dots, k\})$. Then for any other event B for which $P(B) > 0$, the posterior/conditional probability of A_j given that B has occurred is:

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^k P(B|A_i)P(A_i)}$$

$j \in \{1, 2, 3, \dots, k\}$

Derivation

$$\begin{aligned}P(A_j|B) &= \frac{P(A_j \cap B)}{P(B)} \\P(B) &> 0 \\&= \frac{P(B|A_j)P(A_j)}{P(B)} \\&\textit{Multiplication Rule} \\&= \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^k P(B|A_i)P(A_i)} \\&\textit{The Law of Total Probability}\end{aligned}$$

Example

A recent Maryland highway safety study found that in 77% of all accidents the driver was wearing a seatbelt. Accident reports indicated that 92% of those drivers escaped serious injury (defined as hospitalization or death), but only 63% of the non-belted drivers were so fortunate. What is the probability that a driver who was seriously injured wasn't wearing a seatbelt?

Show the steps of finding the answer by defining $B_1, B_2 (= \textit{not } B_1), A_1,$ and A_2 and then using Bayes' Theorem:

$$P(A_2|B_1) = \frac{P(B_1|A_2)P(A_2)}{\sum_{i=1}^2 P(B_1|A_i)P(A_i)}$$

The probability that a driver who was seriously injured wasn't wearing a seatbelt is equal to the probability that a driver wasn't wearing a seatbelt given that he/she was seriously injured.

B_1 : a driver was seriously injured

B_2 : a driver was not seriously injured

A_1 : a driver was wearing a seat belt

A_2 : a driver was not wearing a seat belt

The probability that we want to find is: $P(A_2|B_1)$. A_1 and A_2 are mutually exclusive

and exhaustive.

$$P(A_1) = 0.77$$

$$P(A_2) = 0.23$$

$$P(B_2|A_1) = 0.92$$

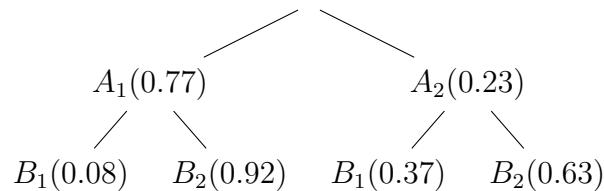
$$P(B_1|A_1) = 0.08$$

$$P(B_2|A_2) = 0.63$$

$$P(B_1|A_2) = 0.37$$

$$\begin{aligned} P(A_2|B_1) &= \frac{P(B_1|A_2)P(A_2)}{P(B_1|A_1)P(A_1) + P(B_1|A_2)P(A_2)} \\ &= \frac{0.37 \times 0.23}{0.08 \times 0.77 + 0.37 \times 0.23} \\ &\approx 0.58 \end{aligned}$$

Show the steps of finding $P(A_2|B_1)$ using a tree diagram.



$$P(A_2|B_1) = \frac{0.23 \times 0.37}{0.77 \times 0.08 + 0.23 \times 0.37} \approx 0.58$$

Example

1 in 1000 adults is afflicted with a rare disease. Consider a test method. How exact is the test? If a person has the disease, there is a 99% chance they will test positive. If a person does not have the disease, there is a 2% chance they will test positive. If a randomly selected individual is tested and the result is positive, what is the probability that the individual has the disease?

By Bayes' Theorem, the probability is 4.7%.

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech