

Linear Algebra

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Review 3

Test Topics:

1. Use Cramer's Rule to solve a 2×2 or 3×3 linear system.

$$x_i = \frac{|A_i(\vec{b})|}{|A|}$$

2. Similar to #45,46 from Section 4.2
3. Similar to Section 4.2: #47-52
4. Similar to one or more exercises from Section 4.2: #53-56, 65, 66
5. Similar to the following from Section 4.4: #36, 37, 42-51, 52a

Example

Solve the system:

$$\begin{aligned}2x - y &= 5 \\ x + 3y &= -1\end{aligned}$$

$$\begin{aligned}
A &= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} & \vec{b} &= \begin{bmatrix} 5 \\ -1 \end{bmatrix} \\
|A_1(\vec{b})| &= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} = 14 \\
|A_2(\vec{b})| &= \begin{vmatrix} 2 & 5 \\ 1 & -1 \end{vmatrix} = -7 \\
|A| &= 6 \\
x_1 &= \frac{14}{6} = 2 \\
x_2 &= \frac{-7}{6} = -1 \\
\vec{x} &= \begin{bmatrix} 2 \\ -1 \end{bmatrix}
\end{aligned}$$

Example

If A and B are invertible matrices, show AB and BA are similar. Find an invertible matrix P such that:

$$P^{-1}ABP = BA$$

$$ABP = PAB$$

$P = A$ satisfies this.

Example

If A and B are similar matrices, show $tr(A) = tr(B)$.

There exists an invertible matrix P such that $P^{-1}AP = B$.

$$\begin{aligned}
tr(B) &= tr(P^{-1}AP) \\
&= tr((P^{-1}A)P) \\
&= tr(P(P^{-1}A)) \\
&= tr(PP^{-1}A) \\
&= tr(IA) \\
&= tr(A)
\end{aligned}$$

Example

Show that if $A \sim B$, then $A^T \sim B^T$.

$$\begin{aligned}
 A \sim B &\rightarrow P^{-1}AP = B \\
 (P^{-1}AP)^T &= B^T \rightarrow P^T A^T (P^{-1})^T = B^T \\
 \text{Let } Q &= (P^{-1})^T \\
 Q^{-1}A^T Q &= B^T \\
 A^T &\sim B^T
 \end{aligned}$$

Example

Prove that if A is diagonalizable, so is A^T . If A is diagonalizable, then $A \sim D$ where D is the diagonal matrix.

If A is diagonalizable, there exists an invertible matrix P such that $P^{-1}AP = D$.

$$\begin{aligned}
 (P^{-1}AP)^T &= D^T \rightarrow P^T A^T (P^{-1})^T = D^T = D \\
 Q &= (P^{-1})^T \\
 Q^{-1}A^T Q &= D \\
 A^T &\sim D
 \end{aligned}$$

Thus, A^T is diagonalizable.

Example

Let A be an invertible matrix. If A is diagonalizable, so is A^{-1} .

$$\begin{aligned}
 P^{-1}AP &= D \\
 (P^{-1}AP)^{-1} &= D^{-1} \\
 P^{-1}A^{-1}(P^{-1})^{-1} &= D^{-1} \\
 P^{-1}A^{-1}P &= D^{-1} \\
 A^{-1} &\sim D^{-1}
 \end{aligned}$$

Therefore, A is diagonalizable.

Example

Prove that if A is a diagonalizable matrix with only 1 eigenvalue λ , then $A = \lambda I$.

$$\begin{aligned} P^{-1}AP &= D \\ &= \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ 0 & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda \end{bmatrix} \\ &= \lambda I \\ A &= P(\lambda I)P^{-1} \\ &= \lambda(PIP^{-1}) \\ &= \lambda(IPP^{-1}) \\ &= \lambda(II) \\ &= \lambda I \end{aligned}$$

Example

Let A, B be similar matrices. Prove that the algebraic multiplicity of the eigenvalues of A, B are the same.

Since A and B are similar, they have the same characteristic polynomial.

Example

Prove that if A is a diagonalizable matrix such that every eigenvalue of A is 0 or 1, then A is idempotent ($A^2 = A$).

$$\begin{aligned} P^{-1}AP &= D \\ A &= PDP^{-1} \\ A^2 &= (PDP^{-1})^2 \\ &= (PDP^{-1})(PDP^{-1}) \\ &= PD^2P^{-1} \\ &= PDP^{-1} \\ &= A \end{aligned}$$

Example

Let A be a nilpotent matrix ($A^m = 0$ for some $m > 1$). Prove that if A is diagonalizable, then $A = 0$.

If A is diagonalizable, there exists an invertible matrix P such that:

$$\begin{aligned}P^{-1}AP &= D \\(P^{-1}AP)^m &= D^m \\P^{-1}A^mP &= D^m \\P^{-1}0P &= D^m \\0 &= D^m \\&= \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ 0 & \vdots & \vdots & 0 \\ 0 & 0 & \dots & \lambda \end{bmatrix} \\ \lambda_1^m &= \dots = \lambda_n^m = 0 \\ D &= 0 \\ P^{-1}AP &= 0 \\ A &= PDP^{-1} \\ &= P0P^{-1} \\ &= 0\end{aligned}$$

Example

Suppose A is a 6×6 matrix with characteristic polynomial $C_A(\lambda) = (1 + \lambda)(1 - \lambda)^2(2 - \lambda)^3$. Prove that we can not find 3 linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ such that $A\vec{v}_1 = 1\vec{v}_1, A\vec{v}_2 = 1\vec{v}_2, A\vec{v}_3 = 1\vec{v}_3$.

The algebraic multiplicity of $\lambda = 1$ is 2. The geometric multiplicity of $\lambda = 1$ is at least 3. This cannot happen since the algebraic multiplicity must upper bound the geometric multiplicity.

If A is diagonalizable, compute the geometric multiplicities of $\lambda = -1, 1, 2$.

$$\begin{aligned}\dim(E_{-1}) &= 1 \\ \dim(E_1) &= 2 \\ \dim(E_2) &= 3\end{aligned}$$

Example

Assume we are working over just the real numbers.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Explain why A is diagonalizable if $(a - d)^2 + 4bc > 0$.

$$\begin{aligned} C_A(\lambda) &= |A - \lambda I| \\ &= \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} \\ &= ad + \lambda(-a - d) + \lambda^2 - bc \\ &= \lambda^2 + \lambda(-a - d) + (ad - bc) \end{aligned}$$

$$\begin{aligned} \text{Quadratic Discriminant } D &= B^2 - 4AC \\ &= (-a - d)^2 - 4(1)(ad - bc) \geq 0 \\ &= (a + d)^2 - 4ad + 4bc \geq 0 \\ &= a^2 + 2ad + d^2 - 4ad + 4bc \geq 0 \\ &= a^2 - 2ad + d^2 + 4bc \geq 0 \\ &= (a - d)^2 + 4bc \geq 0 \end{aligned}$$

Example

Let A, B be similar matrices. Prove that the geometric multiplicities of A and B are the same.

Show that if $B = P^{-1}AP$, then every eigenvector of B is of the form $P^{-1}\vec{v}$ for some eigenvector \vec{v} of A . Let \vec{w} be an eigenvector of B .

$$\begin{aligned} B\vec{w} &= \lambda\vec{w} \rightarrow (P^{-1}AP)(\vec{w}) = \lambda\vec{w} \\ &\rightarrow (AP)\vec{w} = P(\lambda\vec{w}) \\ A(P\vec{w}) &= \lambda(P\vec{w}) \\ \text{Let } \vec{v} &= P\vec{w} \\ A\vec{v} &= \lambda\vec{v} \\ \vec{v} = P\vec{w} &\rightarrow \vec{w} = P^{-1}\vec{v} \end{aligned}$$

Claim: Let $\mathbb{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis for eigenspace E_λ of A . Then $\mathbb{B}' = \{P^{-1}\vec{v}_1, P^{-1}\vec{v}_2, \dots, P^{-1}\vec{v}_k\}$

is a basis for eigenspace E_λ of B . Show that \mathbb{B}' is linearly independent.

$$\begin{aligned}\vec{0} &= \sum_{i=1}^k c_i P^{-1}(\vec{v}_i) \\ &= P^{-1}\left(\sum_{i=1}^k c_i \vec{v}_i\right) \\ \vec{0} &= P\vec{0} \\ &= \sum_{i=1}^k c_i \vec{v}_i \\ c_1 &= c_2 = \dots = c_k = 0\end{aligned}$$

Show $\text{span}(\mathbb{B}') = E_\lambda$ in B . Take $\vec{w} \in E_\lambda$ of B . Then $\vec{w} = P^{-1}\vec{v}$ for some $\vec{v} \in \mathbb{R}^n$.

$$\begin{aligned}\vec{w} &= P^{-1}\left(\sum c_i \vec{v}_i\right) = \sum c_i P^{-1}(\vec{v}_i) \in \text{span}(\mathbb{B}') \\ \text{span}(\mathbb{B}') &= E_\lambda \text{ of } B\end{aligned}$$

Example

Let A, B be $n \times n$ matrices with n distinct eigenvalues. Prove that A and B have the same eigenvectors if and only if $AB = BA$.

Suppose A and B have the same eigenvectors. A, B having n distinct eigenvectors implies that A, B are diagonalizable.

$$\begin{aligned}P^{-1}AP &= D_1 \rightarrow A = PD_1P^{-1} \\ P^{-1}BP &= D_2 \rightarrow B = PD_2P^{-1} \\ AB &= (PD_1P^{-1})(PD_2P^{-1}) = PD_1D_2P^{-1} \\ BA &= (PD_1P^{-1})(PD_2P^{-1}) = PD_2D_1P^{-1}\end{aligned}$$

Since D_1, D_2 are diagonal: $D_1D_2 = D_2D_1$. Assume $AB = BA$. Show that A and B have the same eigenvectors. Take \vec{v} as an eigenvector of A .

$$A(B\vec{v}) = (AB)\vec{v} = B(A\vec{v}) = B(\lambda\vec{v}) = \lambda(B\vec{v})$$

So $B\vec{v}$ is an eigenvector of A .

$$\text{span}(B\vec{v}) = E_\lambda$$

Since we know $B\vec{v} \in \text{span}(B\vec{v})$ and $B\vec{v} = \lambda_i\vec{v}$ for some λ_i . So \vec{v} is an eigenvector of B .

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech