Linear Algebra

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Review 3

Test Topics:

1. Use Cramer's Rule to solve a 2×2 or 3×3 linear system.

$$x_i = \frac{|A_i(\vec{b})|}{|A|}$$

2. Similar to #45,46 from Section 4.2
3. Similar to Section 4.2: #47-52
4. Similar to one or more exercises from Section 4.2: #53-56, 65, 66
5. Similar to the following from Section 4.4: #36, 37, 42-51, 52a

Example

Solve the system:

$$\begin{aligned} 2x - y &= 5 \\ x + 3y &= -1 \end{aligned}$$

$$
\begin{aligned}
A &= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \\
|A_1(\vec{b})| &= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} = 14 \\
|A_2(\vec{b})| &= \begin{vmatrix} 2 & 5 \\ 1 & -1 \end{vmatrix} = -7 \\
|A| &= 6 \\
x_1 &= \frac{14}{7} = 2 \\
x_2 &= \frac{-7}{7} = -1 \\
\vec{x} &= \begin{bmatrix} 2 \\ -1 \end{bmatrix}
\end{aligned}
$$

Example

If A and B are invertible matrices, show AB and BA are similar. Find an invertible matrix P such that:

$$P^{-1}ABP = BA$$

$$ABP = PAB$$

$P = A$ satisfies this.

Example

If A and B are similar matrices, show $tr(A) = tr(B)$.

There exists an invertible matrix P such that $P^{-1}AP = B$.

$$
\begin{aligned}
tr(B) &= tr(P^{-1}AP) \\
&= tr((P^{-1}A)P) \\
&= tr(P(P^{-1}A)) \\
&= tr(PP^{-1}A) \\
&= tr(IA) \\
&= tr(A)
\end{aligned}
$$

Example

Show that if $A \sim B$, then $A^T \sim B^T$.

$$A \sim B \to P^{-1}AP = B$$
$$(P^{-1}AP)^T = B^T \to P^T A^T (P^{-1})^T = B^T$$
$$Let\ Q = (P^{-1})^T$$
$$Q^{-1}A^T Q = B^T$$
$$A^T \sim B^T$$

Example

Prove that if A is diagonalizable, so is A^T. If A is diagonalizable, then $A \sim D$ where D is the diagonal matrix.

If A is diagonalizable, there exists an invertible matrix P such that $P^{-1}AP = D$.

$$(P^{-1}AP)^T = D^T \to P^T A^T (P^{-1})^T = D^T = D$$
$$Q = (P^{-1})^T$$
$$Q^{-1}A^T Q = D$$
$$A^T \sim D$$

Thus, A^T is diagonalizable.

Example

Let A be an invertible matrix. If A is diagonalizable, so is A^{-1}.

$$P^{-1}AP = D$$
$$(P^{-1}AP)^{-1} = D^{-1}$$
$$P^{-1}A^{-1}(P^{-1})^{-1} = D^{-1}$$
$$P^{-1}A^{-1}P = D^{-1}$$
$$A^{-1} \sim D^{-1}$$

Therefore, A is diagonalizable.

Example

Prove that if A is a diagonalizable matrix with only 1 eigenvalue λ, then $A = \lambda I$.

$$\begin{aligned}
P^{-1}AP &= D \\
&= \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ 0 & \vdots & \vdots & 0 \\ 0 & 0 & \dots & \lambda \end{bmatrix} \\
&= \lambda I \\
A &= P(\lambda I)P^{-1} \\
&= \lambda(PIP^{-1}) \\
&= \lambda(IPP^{-1}) \\
&= \lambda(II) \\
&= \lambda I
\end{aligned}$$

Example

Let A, B be similar matrices. Prove that the algebraic multiplicity of the eigenvalues of A, B are the same.

Since A and B are similar, they have the same characteristic polynomial.

Example

Prove that if A is a diagonalizable matrix such that every eigenvalue of A is 0 or 1, then A is idempotent ($A^2 = A$).

$$\begin{aligned}
P^{-1}AP &= D \\
A &= PDP^{-1} \\
A^2 &= (PDP^{-1})^2 \\
&= (PDP^{-1})(PDP^{-1}) \\
&= PD^2P^{-1} \\
&= PDP^{-1} \\
&= A
\end{aligned}$$

Example

Let A be a nilpotent matrix ($A^m = 0$ for some $m > 1$). Prove that if A is diagonalizable, then $A = 0$.

If A is diagonalizable, there exists an invertible matrix P such that:

$$\begin{aligned}
P^{-1}AP &= D \\
(P^{-1}AP)^m &= D^m \\
P^{-1}A^mP &= D^m \\
P^{-1}0P &= D^m \\
0 &= D^m \\
&= \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ 0 & \vdots & \vdots & 0 \\ 0 & 0 & \dots & \lambda \end{bmatrix} \\
\lambda_1^m &= \dots = \lambda_n^m = 0 \\
D &= 0 \\
P^{-1}AP &= 0 \\
A &= PDP^{-1} \\
&= P0P^{-1} \\
&= 0
\end{aligned}$$

Example

Suppose A is a 6×6 matrix with characteristic polynomial $C_A(\lambda) = (1+\lambda)(1-\lambda)^2(2-\lambda)^3$. Prove that we can not find 3 linearly independent vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ such that $A\vec{v_1} = 1\vec{v_1}, A\vec{v_2} = 1\vec{v_2}, A\vec{v_3} = 1\vec{v_3}$.

The algebraic multiplicity of $\lambda = 1$ is 2. The geometric multiplicity of $\lambda = 1$ is at least 3. This cannot happen since the algebraic multiplicity must upper bound the geometric multiplicity.

If A is diagonalizable, compute the geometric multiplicities of $\lambda = -1, 1, 2$.

$$\begin{aligned}
dim(E_{-1}) &= 1 \\
dim(E_1) &= 2 \\
dim(E_2) &= 3
\end{aligned}$$

Example

Assume we are working over just the real numbers.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Explain why A is diagonalizable if $(a-d)^2 + 4bc > 0$.

$$\begin{aligned}
C_A(\lambda) &= |A - \lambda I| \\
&= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \\
&= ad + \lambda(-a-d) + \lambda^2 - bc \\
&= \lambda^2 + \lambda(-a-d) + (ad-bc) \\
\textbf{Quadratic Discriminant } D &= B^2 - 4AC \\
&= (-a-d)^2 - 4(1)(ad-bc) \geq 0 \\
&= (a+d)^2 - 4ad + 4bc \geq 0 \\
&= a^2 + 2ad + d^2 - 4ad + 4bc \geq 0 \\
&= a^2 - 2ad + d^2 + 4bc \geq 0 \\
&= (a-d)^2 + 4bc \geq 0
\end{aligned}$$

Example

Let A, B be similar matrices. Prove that the geometric multiplicities of A and B are the same.

Show that if $B = P^{-1}AP$, then every eigenvector of B is of the form $P^{-1}\vec{v}$ for some eigenvector $\vec{v}$ of A. Let $\vec{w}$ be an eigenvector of B.

$$\begin{aligned}
B\vec{w} = \lambda\vec{w} &\to (P^{-1}AP)(\vec{w}) = \lambda\vec{w} \\
&\to (AP)\vec{w} = P(\lambda\vec{w}) \\
A(P\vec{w}) &= \lambda(P\vec{w}) \\
\textit{Let } \vec{v} &= P\vec{w} \\
A\vec{v} &= \lambda\vec{v} \\
\vec{v} = P\vec{w} &\to \vec{w} = P^{-1}\vec{v}
\end{aligned}$$

Claim: Let $\mathbb{B}\{\vec{v_1}, \ldots, \vec{v_k}\}$ be a basis for eigenspace E_λ of A. Then $\mathbb{B}' = \{P^{-1}\vec{v_1}, P^{-1}\vec{v_2}, \ldots, P^{-1}\vec{v_k}\}$

is a basis for eigenspace E_λ of B. Show that $\mathbb{B}'$ is linearly independent.

$$\begin{aligned}\vec{0} &= \sum_{i=1}^{k} c_i P^{-1}(\vec{v_i}) \\ &= P^{-1}(\sum_{i=1}^{k} c_i \vec{v_i}) \\ \vec{0} &= P\vec{0} \\ &= \sum_{i=1}^{k} c_i \vec{v_i} \\ c_1 &= c_2 = \cdots = c_k = 0\end{aligned}$$

Show $span(\mathbb{B}') = E_\lambda$ in B. Take $\vec{w} \in E_\lambda$ of B. Then $\vec{w} = P^{-1}\vec{v}$ for some $\vec{v} \in \mathbb{R}^n$.

$$\vec{w} = P^{-1}(\sum c_i \vec{v_i}) = \sum c_1 P^{-1}(\vec{v_i}) \in span(\mathbb{B}')$$
$$span(\mathbb{B}') = E_\lambda \ of \ B$$

Example

Let A, B be $n \times n$ matrices with n distinct eigenvalues. Prove that A and B have the same eigenvectors if and only if $AB = BA$.

Suppose A and B have the same eigenvectors. A, B having n distinct eigenvectors implies that A, B are diagonalizable.

$$P^{-1}AP = D_1 \to A = PD_1P^{-1}$$
$$P^{-1}BP = D_2 \to B = PD_2P^{-1}$$
$$AB = (PD_1P^{-1})(PD_2P^{-1}) = PD_1D_2P^{-1}$$
$$BA = (PD_1P^{-1})(PD_1P^{-1}) = PD_2D_1P^{-1}$$

Since D_1, D_2 are diagonal: $D_1D_2 = D_2D_1$. Assume $AB = BA$. Show that A and B have the same eigenvectors. Take $\vec{v}$ as an eigenvector of A.

$$A(B\vec{v}) = (AB)\vec{v} = B(A\vec{v}) = B(\lambda\vec{v}) = \lambda(B\vec{v})$$

So $B\vec{v}$ is an eigenvector of A.

$$span(\vec{v}) = E_\lambda$$

Since we know $B\vec{v} \in span(\vec{v})$ and $B\vec{v} = \lambda_i\vec{v}$ for some λ_i. So $\vec{v}$ is an eigenvector of B.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech