Linear Algebra: Homework 8

Alvin Lin

August 2016 - December 2016

Section 4.1

Exercise 1

Show that $\vec{v}$ is an eigenvector of A and find the corresponding eigenvalue.

$$\begin{aligned} A &= \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \\ \vec{v} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ A\vec{v} &= \begin{bmatrix} 3 \\ 3 \end{bmatrix} \\ &= 3\vec{v} \\ \lambda &= 3 \end{aligned}$$

Exercise 3

Show that $\vec{v}$ is an eigenvector of A and find the corresponding eigenvalue.

$$\begin{aligned} A &= \begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix} \\ \vec{v} &= \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ A\vec{v} &= \begin{bmatrix} -3 \\ 6 \end{bmatrix} \\ &= -3\vec{v} \\ \lambda &= -3 \end{aligned}$$

Exercise 5

Show that $\vec{v}$ is an eigenvector of A and find the corresponding eigenvalue.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$
$$\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$
$$A\vec{v} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$$
$$= 3\vec{v}$$
$$\lambda = 3$$

Exercise 7

Show that λ is an eigenvalue of A and find one eigenvector corresponding to this eigenvalue.

$$A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$
$$\lambda = 3$$
$$A - \lambda I = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$$
$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right] = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$x_1 = 2x_2$$
$$\vec{x} = span\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

One possible eigenvector is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Exercise 9

Show that λ is an eigenvalue of A and find one eigenvector corresponding to this eigenvalue.

$$A = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix}$$
$$\lambda = 1$$
$$A - \lambda I = \begin{bmatrix} -1 & 4 \\ -1 & 4 \end{bmatrix}$$
$$\left[\begin{array}{cc|c} -1 & 4 & 0 \\ -1 & 4 & 0 \end{array}\right] = \begin{bmatrix} -1 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$4x_2 = x_1$$
$$\vec{x} = span\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right)$$

One possible eigenvector is $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Exercise 11

Show that λ is an eigenvalue of A and find one eigenvector corresponding to this eigenvalue.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\lambda = -1$$

$$A - \lambda I = \begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -x_3 = x_2$$

$$\vec{x} = span\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)$$

One possible eigenvector is $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

Exercise 27

Find all of the eigenvalues of the matrix A over the complex numbers $\mathbb{C}$. Give bases for each of the corresponding eigenspaces.

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)(1-\lambda) + 1 \\ &= 1 - 2\lambda + \lambda^2 + 1 \\ &= \lambda^2 - 2\lambda + 2 = 0 \end{aligned}$$

$$\begin{aligned} \lambda &= \frac{2 \pm \sqrt{4-8}}{2} \\ &= 1 \pm i \end{aligned}$$

$$A - (1+i)I = \begin{bmatrix} -i & 1 \\ -1 & i \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] = \begin{bmatrix} -i & 1 & 0 \\ -i & 1 & 0 \end{bmatrix}$$

$$x_2 = ix_1$$

$$E_{1+i} = span\left(\begin{bmatrix} i \\ 1 \end{bmatrix}\right)$$

$$A - (1-i)I = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$$

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] = \begin{bmatrix} i & 1 & 0 \\ -i & -1 & 0 \end{bmatrix}$$

$$ix_1 = -1x_2$$

$$E_{1-i} = span\left(\begin{bmatrix} -i \\ 1 \end{bmatrix}\right)$$

Exercise 29

Find all of the eigenvalues of the matrix A over the complex numbers $\mathbb{C}$. Give bases for each of the corresponding eigenspaces.

$$A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & i \\ i & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2 - (-1)$$

$$= 1 - 2\lambda + \lambda^2 + 1 = 0$$

$$\lambda = 1 \pm i$$

$$A - (1+i)I = \begin{bmatrix} -i & i \\ i & -i \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -i & i & 0 \\ i & -i & 0 \end{array}\right] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_2$$

$$E_{1+i} = span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$A - (1-i)I = \begin{bmatrix} i & i \\ i & i \end{bmatrix}$$

$$\left[\begin{array}{cc|c} i & i & 0 \\ i & i & 0 \end{array}\right] = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -x_2$$

$$E_{1-i} = span\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$

Section 4.2

Exercise 1

Compute the determinant using cofactor expansion along the first row and column.

$$\begin{aligned}
\begin{vmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} &= 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 0 + 3\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} \\
&= 1(2-1) + 3(5) \\
&= 16 \\
&= 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 5\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} + 0 \\
&= 1(2-1) - 5(0-3) \\
&= 16
\end{aligned}$$

Exercise 3

Compute the determinant using cofactor expansion along the first row and column.

$$\begin{aligned}
\begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} &= 1\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} + 0 \\
&= 1(-1) + 1(1) \\
&= 0
\end{aligned}$$

Since the matrix is symmetric, the expansion is the same along the first row and column.

Exercise 5

Compute the determinant using cofactor expansion along the first row and column.

$$\begin{aligned}
\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} &= 1\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} \\
&= 1(6-1) - 2(4-3) + 3(2-9) \\
&= 5 - 2 - 21 \\
&= -18
\end{aligned}$$

Since the matrix is symmetric, the expansion is the same along the first row and column.

Exercise 7

Compute the determinant using cofactor expansion along any row or column that seems convenient.

$$\begin{aligned}
\begin{vmatrix} 5 & 2 & 2 \\ -1 & 1 & 2 \\ 3 & 0 & 0 \end{vmatrix} &= 3\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} - 0 + 0 \\
&= 3(4-2) \\
&= 6
\end{aligned}$$

Exercise 9

Compute the determinant using cofactor expansion along any row or column that seems convenient.

$$\begin{vmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix} - (-1)\begin{vmatrix} -4 & 3 \\ 2 & 4 \end{vmatrix} + 0$$
$$= 1(4+6) + 1(-16-6)$$
$$= -12$$

Exercise 11

Compute the determinant using cofactor expansion along any row or column that seems convenient.

$$\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ a & 0 & b \end{vmatrix} = a\begin{vmatrix} a & b \\ 0 & b \end{vmatrix} - b\begin{vmatrix} 0 & b \\ a & b \end{vmatrix} + 0$$
$$= a(ab) - b(-ba)$$
$$= a^2b + ab^2$$
$$= ab(a+b)$$

Exercise 13

Compute the determinant using cofactor expansion along any row or column that seems convenient.

$$\begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{vmatrix} = 0 - 1\begin{vmatrix} 1 & 0 & 3 \\ 2 & 2 & 6 \\ 1 & 2 & 1 \end{vmatrix} + 0 - 0$$
$$= -1\left(1\begin{vmatrix} 2 & 6 \\ 2 & 1 \end{vmatrix} - 0 + 3\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}\right)$$
$$= -1(1(2-12) + 3(4-2))$$
$$= 4$$

Exercise 15

Compute the determinant using cofactor expansion along any row or column that seems convenient.

$$\begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & c \\ 0 & d & e & f \\ g & h & i & j \end{vmatrix} = 0 - 0 + 0 - a\begin{vmatrix} 0 & 0 & b \\ 0 & d & e \\ g & h & i \end{vmatrix}$$
$$= -a(0 - 0 + b\begin{vmatrix} 0 & d \\ g & h \end{vmatrix})$$
$$= -ab(-dg)$$
$$= abdg$$

Exercise 21

Prove Theorem 4.2: The determinant of a triangular matrix is the product of the entries on its main diagonal. Specifically, if $A = [a_{ij}]$ is an $n \times n$ triangular matrix, then

$$det(A) = a_{11}a_{22}\cdots a_{nn}$$

Base Case:

$$|a_{11}| = a_{11}$$

Induction Hypothesis:

$$det(A) = a_{11}a_{22}a_{33}\cdots a_{nn}$$

Induction:

$$\begin{vmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & 0 \\ a_{(n+1)1} & a_{(n+1)2} & \dots & a_{(n+1)(n+1)} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & \dots & 0 \\ \vdots & \vdots & 0 \\ a_{(n+1)2} & \dots & a_{(n+1)(n+1)} \end{vmatrix}$$
$$= a_{11}a_{22}a_{33}\cdots a_{nn}a_{(n+1)(n+1)}$$

Exercise 23

Evaluate the given determinant using elementary row and/or column operations and Theorem 4.3 to reduce the matrix to row echelon form.

$$A = \begin{bmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{bmatrix} (\frac{1}{2}R_2 \to R_2)$$
$$= \begin{bmatrix} -4 & 1 & 3 \\ 1 & -1 & 2 \\ 1 & -1 & 0 \end{bmatrix} (R_3 - R_2 \to R_3) \quad (4R_2 \to R_2)$$
$$= \begin{bmatrix} -4 & 1 & 3 \\ 4 & -4 & 8 \\ 0 & 0 & -2 \end{bmatrix} (R_2 + R_1 \to R_2)$$
$$= \begin{bmatrix} -4 & 1 & 3 \\ 0 & -3 & 11 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\left(\frac{1}{2}\right)(4)det(A) = (-4)(-3)(-2)$$
$$det(A) = -12$$

Exercise 25

Evaluate the given determinant using elementary row and/or column operations and Theorem 4.3 to reduce the matrix to row echelon form.

$$A = \begin{bmatrix} 2 & 0 & 3 & -1 \\ 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{bmatrix} (R_1 - 2R_2 \to R_1)$$
$$= \begin{bmatrix} 0 & 0 & -1 & -5 \\ 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{bmatrix} (R_4 - 2R_2 \to R_2)$$
$$= \begin{bmatrix} 0 & 0 & -1 & -5 \\ 0 & 0 & -3 & -7 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{bmatrix} (R_2 - 3R_1 \to R_1)$$
$$= \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & -3 & -7 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{bmatrix}$$
$$(-2)(-3)det(A) = (2)(-1)(-3)(8)$$
$$det(A) = 8$$

Exercise 27

Use properties of determinants to evaluate the given determinant by inspection. Explain your reasoning.

$$\begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 4 \end{vmatrix} = -24$$

The matrix is upper triangular. The determinant of a triangular matrix is the product of the elements on the diagonal.

Exercise 29

Use properties of determinants to evaluate the given determinant by inspection. Explain your reasoning.

$$\begin{vmatrix} 2 & 3 & -4 \\ 1 & -3 & -2 \\ -1 & 5 & 2 \end{vmatrix} = 0$$

Multiplying column 1 by -2 would makes it identical to column 3.

Exercise 31

Use properties of determinants to evaluate the given determinant by inspection. Explain your reasoning.

$$\begin{vmatrix} 4 & 1 & 3 \\ -2 & 0 & -2 \\ 5 & 4 & 1 \end{vmatrix} = 0$$

Subtracting the second column from the first column makes the first column identical to the third column.

Exercise 33

Use properties of determinants to evaluate the given determinant by inspection. Explain your reasoning.

$$\begin{vmatrix} 0 & 2 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -24$$

Swapping the first and second rows and swapping the third and fourth rows makes it a triangular matrix. These two operations negate the determinant and cancel out. The determinant of a triangular matrix is the product of the elements on the diagonal.

Exercise 35

Find the determinants assuming that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4$$

$$\begin{vmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{vmatrix} = 8$$

Exercise 37

Find the determinants assuming that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4$$

$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = -4$$

Exercise 39

Find the determinants assuming that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4$$

$$\begin{vmatrix} 2c & b & a \\ 2f & e & d \\ 2i & h & g \end{vmatrix} = -8$$

Exercise 45

Use Theorem 4.6 to find all values of k for which A is invertible.

$$A = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix}$$

$$\begin{aligned} 0 &\neq \begin{vmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{vmatrix} \\ &\neq k \begin{vmatrix} k+1 & 1 \\ -8 & k-1 \end{vmatrix} - 0 + k \begin{vmatrix} -k & 3 \\ k+1 & 1 \end{vmatrix} \\ &\neq k(k^2 - 1 - (-8)) + k(-k - (3k+3)) \\ &\neq k^3 + 7k - k^2 - 3k^2 - 3k \\ &\neq k^3 - 4k^2 + 4k \\ &\neq k(k^2 - 4k + 4) \\ &\neq k(k-2)(k-2) \\ k &\neq 0 \quad k \neq 2 \end{aligned}$$

Exercise 46

Use Theorem 4.6 to find all values of k for which A is invertible.

$$A = \begin{bmatrix} k & k & 0 \\ k^2 & 2 & k \\ 0 & k & k \end{bmatrix}$$

$$\begin{aligned} 0 &\neq \begin{vmatrix} k & k & 0 \\ k^2 & 2 & k \\ 0 & k & k \end{vmatrix} \\ &\neq k \begin{vmatrix} 2 & k \\ k & k \end{vmatrix} - k \begin{vmatrix} k^2 & k \\ 0 & k \end{vmatrix} + 0 \\ &\neq k(2k - k^2) - k(k^3) \\ &\neq 2k^2 - k^3 - k^4 \\ &\neq -k^2(k^2 + k - 2) \\ &\neq -k^2(k+2)(k-1) \\ k &\neq 0 \quad k \neq -2 \quad k \neq 1 \end{aligned}$$

Exercise 47

Assume that A and B are $n \times n$ matrices with $det(A) = 3$ and $det(B) = -2$. Find the indicated determinants.

$$det(AB) = -6$$

Exercise 48

Assume that A and B are $n \times n$ matrices with $det(A) = 3$ and $det(B) = -2$. Find the indicated determinants.

$$det(A^2) = 9$$

Exercise 49

Assume that A and B are $n \times n$ matrices with $det(A) = 3$ and $det(B) = -2$. Find the indicated determinants.

$$det(B^{-1}A) = \frac{1}{-2}3 = -\frac{3}{2}$$

Exercise 50

Assume that A and B are $n \times n$ matrices with $det(A) = 3$ and $det(B) = -2$. Find the indicated determinants.

$$det(2A) = 2^n(3)$$

Exercise 51

Assume that A and B are $n \times n$ matrices with $det(A) = 3$ and $det(B) = -2$. Find the indicated determinants.

$$det(3B^T) = 3^n(-2)$$

Exercise 52

Assume that A and B are $n \times n$ matrices with $det(A) = 3$ and $det(B) = -2$. Find the indicated determinants.

$$det(AA^T) = 9$$

Exercise 57

Use Cramer's Rule to solve the given linear system.

$$\begin{aligned}
x + y &= 1 \\
x - y &= 2 \\
A &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
det(A) &= -2 \\
det(A_1\vec{b}) &= \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3 \\
det(A_2\vec{b}) &= \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \\
x &= \frac{-3}{-2} = \frac{3}{2} \\
y &= \frac{1}{-2} = -\frac{1}{2}
\end{aligned}$$

Exercise 58

Use Cramer's Rule to solve the given linear system.

$$2x - y = 5$$
$$x + 3y = -1$$
$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$
$$det(A) = 7$$
$$det(A_1\vec{b}) = \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} = 14$$
$$det(A_2\vec{b}) = \begin{vmatrix} 2 & 5 \\ 1 & -1 \end{vmatrix} = -7$$
$$x = \frac{14}{7} = 2$$
$$y = \frac{-7}{7} = -1$$

Exercise 59

Use Cramer's Rule to solve the given linear system.

$$2x + y + 3z = 1$$
$$y + z = 1$$
$$z = 1$$
$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$det(A) = 2$$
$$det(A_1\vec{b}) = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = -2$$
$$det(A_2\vec{b}) = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$
$$det(A_3\vec{b}) = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2$$
$$x = \frac{-2}{2} = -1$$
$$y = \frac{0}{2} = 0$$
$$z = 1$$

Exercise 60

Use Cramer's Rule to solve the given linear system.

$$\begin{aligned}
x + y - z &= 1 \\
x + y + z &= 2 \\
x - y &= 3 \\
A &= \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
det(A) &= 2 + 2 = 4 \\
det(A_1\vec{b}) &= \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{vmatrix} = 6 + 3 = 9 \\
det(A_2\vec{b}) &= \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{vmatrix} = 3 - 6 = -3 \\
det(A_3\vec{b}) &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & -1 & 3 \end{vmatrix} = 1 + 1 + 0 = 2 \\
x &= \frac{9}{4} \\
y &= \frac{-3}{4} \\
z &= \frac{2}{4}
\end{aligned}$$

Exercise 61

Compute the inverse of the coefficient matrix.

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
C_{11} &= -1 \\
C_{12} &= -1 \\
C_{21} &= -1 \\
C_{22} &= 1 \\
A^{-1} &= \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}
\end{aligned}$$

Exercise 62

Compute the inverse of the coefficient matrix.

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$
$$C_{11} = 3$$
$$C_{12} = -1$$
$$C_{21} = 1$$
$$C_{22} = 2$$
$$A^{-1} = \frac{1}{7}\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}^T$$
$$= \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

Exercise 63

Compute the inverse of the coefficient matrix.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
$$adj(A) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -2 & -2 & 2 \end{bmatrix}^T$$
$$\frac{1}{det(A)} = \frac{1}{2}$$
$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 64

Compute the inverse of the coefficient matrix.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$
$$adj(A) = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 2 \\ 2 & -2 & 0 \end{bmatrix}^T$$
$$\frac{1}{det(A)} = \frac{1}{4}$$
$$A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Section 4.3

Exercise 1

Compute the characteristic polynomial of A, the eigenvalues of A, a basis for each eigenspace of A, and the algebraic and geometric multiplicity of each eigenvalue.

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ -2 & 6-\lambda \end{vmatrix} = (6-\lambda)(1-\lambda) - (-6)$$

$$= 6 - 7\lambda + \lambda^2 + 6 = \lambda^2 - 7\lambda + 12$$

$$= (\lambda - 3)(\lambda - 4) = 0$$

$$\lambda = 3 \quad \lambda = 4$$

$$[A - 3I|0] = \begin{bmatrix} -2 & 3 & 0 \\ -2 & 3 & 0 \end{bmatrix}$$

$$3x_2 = 2x_1$$

$$E_3 = span\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)$$

$$[A - 4I|0] = \left[\begin{array}{cc|c} -3 & 3 & 0 \\ -2 & 2 & 0 \end{array}\right] = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_2$$

$$E_4 = span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

Each eigenvalue has algebraic and geometric multiplicity 1.

Exercise 3

Compute the characteristic polynomial of A, the eigenvalues of A, a basis for each eigenspace of A, and the algebraic and geometric multiplicity of each eigenvalue.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda)(3-\lambda) = 0$$

$$\lambda = 1 \quad \lambda = -2 \quad \lambda = 3$$

$$[A - 1I|0] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_2 = x_3 = 0$$

$$E_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[A+2I|0] = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$3x_1 = -x_2$$

$$x_3 = 0$$

$$E_{-2} = span\left(\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}\right)$$

$$[A-3I|0] = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 & -1 & 0 \\ 0 & 5 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$10x_1 = 5x_2 = x_3$$

$$E_3 = span\left(\begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix}\right)$$

The algebraic and geometric multiplicities of the eigenvalues are 1.

Exercise 5

Compute the characteristic polynomial of A, the eigenvalues of A, a basis for each eigenspace of A, and the algebraic and geometric multiplicity of each eigenvalue.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)((1-\lambda)(-1-\lambda)-1) - 2(-(1-\lambda))$$

$$= (1-\lambda)(-1+\lambda^2-1) + 2 - 2\lambda$$

$$= -2 + \lambda^2 + 2\lambda - \lambda^3 + 2 - 2\lambda$$

$$= -\lambda^3 + \lambda^2$$

$$= -\lambda^2(\lambda - 1) = 0$$

$$\lambda = 0 \quad \lambda = 1$$

$$[A-0I|0] = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 2x_3$$

$$x_2 = -x_3$$

$$E_0 = span\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right)$$

$$[A - 1I|0] = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_3$$

$$x_2 = 0$$

$$E_1 = span\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

$\lambda = 0$ has algebraic multiplicity 2 while $\lambda = 1$ has algebraic multiplicity 1. They both have geometric multiplicity 1.

Exercise 7

Compute the characteristic polynomial of A, the eigenvalues of A, a basis for each eigenspace of A, and the algebraic and geometric multiplicity of each eigenvalue.

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ -1 & 0 & 2-\lambda \end{vmatrix} = (4-\lambda)((3-\lambda)(2-\lambda)) + (0 + (3-\lambda))$$

$$= (4-\lambda)(6 - 5\lambda + \lambda^2) + 3 - \lambda$$

$$= 24 - 20\lambda + 4\lambda^2 - 6\lambda + 5\lambda^2 - \lambda^3 + 3 - \lambda$$

$$= -\lambda^3 + 9\lambda^2 - 27\lambda + 27 = 0$$

$$\lambda = 3$$

$$[A - 3I|0] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -x_3$$

$$E_3 = \begin{bmatrix} x_1 & x_2 & -x_1 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= span\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$\lambda = 3$ has algebraic multiplicity 3 and geometric multiplicity 2.

Exercise 9

Compute the characteristic polynomial of A, the eigenvalues of A, a basis for each eigenspace of A, and the algebraic and geometric multiplicity of each eigenvalue.

$$A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned}
|A - \lambda I| &= \begin{vmatrix} 3-\lambda & 1 & 0 & 0 \\ -1 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 4 \\ 0 & 0 & 1 & 1-\lambda \end{vmatrix} \\
&= ((3-\lambda)(1-\lambda)+1)((1-\lambda)^2-4) \\
&= \lambda^4 - 6\lambda^3 + 9\lambda^2 + 4\lambda - 12 \\
&= \lambda^2(\lambda^2 - 6\lambda + 9) + 4(\lambda - 3) \\
&= (\lambda - 3)(\lambda^3 - 3\lambda^2 + 4) \\
&= (\lambda - 3)(\lambda - 2)^2(\lambda + 1) = 0 \\
\lambda &= -1 \quad \lambda = 2 \quad \lambda = 3
\end{aligned}$$

$$[A + 1I|0] = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_2 = 0$$

$$x_3 = -2x_4$$

$$E_{-1} = span\left(\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right)$$

$$[A - 2I|0] = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -x_2$$

$$x_3 = x_4 = 0$$

$$E_2 = span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$[A-3I|0] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_2 = 0$$

$$x_3 = 2x_4$$

$$E_3 = span\left(\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right)$$

Exercise 11

Compute the characteristic polynomial of A, the eigenvalues of A, a basis for each eigenspace of A, and the algebraic and geometric multiplicity of each eigenvalue.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ -2 & 1 & 2 & -1 \end{bmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 1 & 1 & 3-\lambda & 0 \\ -2 & 1 & 2 & -1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(1-\lambda)(3-\lambda)(-1-\lambda) = 0$$

$$\lambda = 1 \quad \lambda = 3 \quad \lambda = -1$$

$$[A-1I|0] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ -2 & 1 & 2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 3 & 0 & 0 & 2 & 0 \end{bmatrix}$$

$$x_1 + x_2 + 2x_3 = 0$$

$$3x_1 + 2x_4 = 0$$

$$E_1 = \begin{bmatrix} x_1 \\ -2x_3 - x_1 \\ x_3 \\ \frac{-3x_1}{2} \end{bmatrix} = \begin{bmatrix} 2x_1 \\ -4x_3 - 2x_1 \\ 2x_3 \\ -3x_1 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ -2 \\ 0 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -4 \\ 2 \\ 0 \end{bmatrix}$$

$$= span\left(\begin{bmatrix} 2 \\ -2 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}\right)$$

Exercise 13

Prove Theorem 4.18b: Let A be a square matrix with eigenvalue λ and corresponding eigenvector $\vec{x}$. If A is invertible, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with corresponding eigenvector $\vec{x}$.

$$\begin{aligned} A\vec{x} &= \lambda\vec{x} \\ A^{-1}A\vec{x} &= A^{-1}(\lambda\vec{x}) \\ \vec{x} &= \lambda A^{-1}\vec{x} \\ A^{-1}\vec{x} &= \frac{1}{\lambda}\vec{x} \end{aligned}$$

Exercise 14

Prove Theorem 4.18c: Let A be a square matrix with eigenvalue λ and corresponding eigenvector $\vec{x}$. If A is invertible, then for any integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector $\vec{x}$.
Base Case $n = 1$:

$$A^1\vec{x} = \lambda^1\vec{x}$$

Induction Hypothesis: $A^n\vec{x} = \lambda^n\vec{x}$
Induction Step:

$$\begin{aligned} A^{n+1}\vec{x} &= A(A^n\vec{x}) \\ &= A(\lambda^n\vec{x}) \\ &= \lambda^n(A\vec{x}) \\ &= \lambda^n(\lambda\vec{x}) \\ &= \lambda^{n+1}\vec{x} \end{aligned}$$

Exercise 20

Let A be a nilpotent matrix. Show that $\lambda = 0$ is the only eigenvalue of A.

$$|A^m| = |0| = 0$$

Thus $\lambda = 0$ is an eigenvalue. Show that there are no other eigenvalues.
Suppose λ is another eigenvalue ($\lambda \neq 0$). Then λ^m is an eigenvalue of A^m. This forces $\lambda^m = 0 \therefore \lambda = 0$.

Exercise 21

Let A be an idempotent matrix. Show that $\lambda = 0$ and $\lambda = 1$ are the only possible eigenvalues of A.
Since A is idempotent, it means that $A^2 = A$. Let λ be an eigenvalue of A, then λ^2 is an eigenvalue of $A^2 = A$.

$$\begin{aligned} \lambda^2 &= \lambda \\ \lambda^2 - \lambda &= 0 \\ \lambda(\lambda - 1) &= 0 \\ \lambda &= 0 \quad \lambda = 1 \end{aligned}$$

Exercise 22

If $\vec{v}$ is an eigenvector of A with corresponding eigenvalue λ and c is a scalar, show that $\vec{v}$ is an eigenvector of $A - cI$ with corresponding eigenvalue $\lambda - c$.

$$\begin{aligned} B &= A - cI \\ (A - \lambda I)\vec{v} &= 0 \\ B - (\lambda - c)I &= A - cI - (\lambda - c)I \\ &= A - cI - \lambda I + cI \\ &= A - \lambda I \\ (B - (\lambda - c)I)\vec{v} &= (A - \lambda I)\vec{v} = 0 \end{aligned}$$

Section 4.4

Exercise 1

Show that A and B are not similar matrices.

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 4-\lambda & 1 \\ 3 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) - 3 \\ &= 4 - 5\lambda + \lambda^2 - 3 \\ &= \lambda^2 - 5\lambda - 7 \\ |B - \lambda I| &= \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) \end{aligned}$$

Since A and B do not have the same characteristic polynomial, they are not similar.

Exercise 3

Show that A and B are not similar matrices.

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 2 & 3 & 4 \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= (2-\lambda)(2-\lambda)(4-\lambda) \\ |B - \lambda I| &= (1-\lambda)(4-\lambda)(4-\lambda) \end{aligned}$$

Since A and B do not have the same characteristic polynomial, they are not similar.

Exercise 5

A diagonalization of the matrix A is given in form $P^{-1}AP = D$. List the eigenvalues of A and bases from the corresponding eigenspaces.

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\lambda = 4 \quad \lambda = 3$$

$$E_4 = span\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$$

$$E_3 = span\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$$

Exercise 7

A diagonalization of the matrix A is given in form $P^{-1}AP = D$. List the eigenvalues of A and bases from the corresponding eigenspaces.

$$\begin{bmatrix}\frac{1}{8} & \frac{1}{8} & \frac{1}{8}\\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4}\\ \frac{5}{8} & -\frac{3}{8} & -\frac{3}{8}\end{bmatrix}\begin{bmatrix}1 & 3 & 3\\ 2 & 0 & 2\\ 3 & 3 & 1\end{bmatrix}\begin{bmatrix}3 & 0 & 1\\ 2 & 1 & 0\\ 3 & -1 & -1\end{bmatrix} = \begin{bmatrix}6 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -2\end{bmatrix}$$

$$\lambda = 6 \quad \lambda = -2 \quad \lambda = -2$$

$$E_6 = span\left(\begin{bmatrix}3\\2\\3\end{bmatrix}\right)$$

$$E_{-2} = span\left(\begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right)$$

Exercise 9

Determine whether A is diagonalizable and, if so, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$A = \begin{bmatrix}-3 & 4\\ -1 & 1\end{bmatrix}$$

$$\begin{aligned}|A - \lambda I| = \begin{vmatrix}-3-\lambda & 4\\ -1 & 1-\lambda\end{vmatrix} &= (-3-\lambda)(1-\lambda) + 4\\ &= -3 + 2\lambda + \lambda^2 + 4 = \lambda^2 + 2\lambda + 1\\ &= (\lambda + 1)^2 = 0\end{aligned}$$

A is not diagonalizable since there cannot be two linearly independent eigenvectors.

Exercise 11

Determine whether A is diagonalizable and, if so, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 1 & \\ 0 & 1-\lambda & 1 & 1 \\ 1 & 1 & -\lambda & \end{vmatrix} = -\lambda^3 + 2\lambda^2 + \lambda - 2$$

$$= -\lambda^2(\lambda - 2) + 1(\lambda - 2)$$

$$= (1-\lambda)(1+\lambda)(\lambda - 2) = 0$$

$$\lambda = 1 \quad \lambda = -1 \quad \lambda = 2$$

$$[A - 1I|0] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

$$E_1 = span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$[A + 1I|0] = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$E_{-1} = span\left(\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}\right)$$

$$[A - 2I|0] = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$E_2 = span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

The algebraic multiplicities of the eigenvectors are equal to their geometric multiplicities, so the matrix is diagonalizable. The following matrices P and D satisfy $P^{-1}AP = D$.

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Exercise 13

Determine whether A is diagonalizable and, if so, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 1 \\ -1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = \lambda^2(\lambda - 1) = 0$$

$$\lambda = 0 \quad \lambda = 1$$

$$[A - 0I|0] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$E_0 = span\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)$$

Since the geometric multiplicity is not equal to the algebraic multiplicity for $\lambda = 0$, this matrix is not diagonalizable.

Exercise 15

Determine whether A is diagonalizable and, if so, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$A = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$\lambda = 2 \quad \lambda = 2$$

$$[A - 2I|0] = \begin{bmatrix} 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_3 = x_4 = 0$$

$$E_2 = span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$[A+2I|0] = \begin{bmatrix} 4 & 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -x_4$$
$$x_2 = 0$$

$$E_{-2} = \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

Since the algebraic multiplicities of the eigenvalues is equal to their respective geometric multiplicities, the matrix is diagonalizable. The following matrices P and D satisfy $P^{-1}AP = D$.

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Exercise 17

Use the method of Example 4.29 to compute the indicated power of the matrix.

$$A = \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 6 \\ 1 & -\lambda \end{vmatrix}$$
$$= -\lambda(-1-\lambda) - 6 = \lambda^2 + \lambda - 6$$
$$= (\lambda+3)(\lambda-2) = 0$$
$$\lambda = -3 \quad \lambda = 2$$

$$[A+3I|0] = \begin{bmatrix} 2 & 6 & 0 \\ 1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_{-3} = span\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)$$

$$[A - 2I|0] = \begin{bmatrix} -3 & 6 & 0 \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_2 = span\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$$

$$P^{-1} = -1\begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix}$$

$$P^{-1}AP = D$$

$$P^{-1}A^{10}P = D^{10}$$

$$A^{10} = PD^{10}P^{-1}$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2^{10} & 0 \\ 0 & (-3)^{10} \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$$

Exercise 19

Use the method of Example 4.29 to compute the indicated power of the matrix.

$$A = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = -\lambda(2-\lambda) - 3$$

$$= \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$$

$$\lambda = 3 \quad \lambda = -2$$

$$[A - 3I|0] = \begin{bmatrix} -3 & 3 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_3 = span\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

$$[A + 1I|0] = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

$$E_{-2} = span\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)$$

$$
\begin{aligned}
D &= \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \\
P &= \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix} \\
P^{-1} &= \frac{1}{2}\begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix} \\
P^{-1}AP &= D \\
P^{-1}A^kP &= D^k \\
A^k &= PD^kP^{-1} \\
&= \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & (-2)^k \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}
\end{aligned}
$$

Exercise 21

Use the method of Example 4.29 to compute the indicated power of the matrix.

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
|A - \lambda I| &= (1-\lambda)(-1-\lambda)(-1-\lambda) = 0 \\
\lambda &= -1 \quad \lambda = 1 \\
[A - 1I|0] &= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
E_1 &= span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \\
[A + 1I|0] &= \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
E_{-1} &= \begin{bmatrix} x_1 \\ x_2 \\ -x_2 - 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \\
&= span\left(\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)
\end{aligned}
$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^{2015} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1^{2015} & 0 & 0 \\ 0 & (-1)^{2015} & 0 \\ 0 & 0 & (-1)^{2015} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

Exercise 23

Use the method of Example 4.29 to compute the indicated power of the matrix.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned}
|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 2 & -2-\lambda & 2 \\ 0 & 1 & 1-\lambda \end{vmatrix} &= (1-\lambda)((-2-\lambda)(1-\lambda) - 2) - 2(1-\lambda) \\
&= (1-\lambda)(\lambda^2 + \lambda - 4) - 2 + 2\lambda \\
&= \lambda^2 + \lambda - 4 - \lambda^3 - \lambda^2 + 4\lambda - 2 + 2\lambda \\
&= -\lambda^3 + 7\lambda - 6 = 0
\end{aligned}$$

$$\lambda = -3 \quad \lambda = 1 \quad \lambda = 2$$

$$[A + 3I|0] = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_{-3} = span\left(\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}\right)$$

$$[A - 1I|0] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & -3 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_1 = span\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$$

$$[A - 2I|0] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_2 = span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

$$D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$A^k = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} (-3)^k & 0 & 0 \\ 0 & 1^k & 0 \\ 0 & 0 & 2^k \end{bmatrix} \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Exercise 25

Find all real values of k for which A is diagonalizable.

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$
$$|A - \lambda I| = (1 - \lambda)^2 = 0$$
$$\lambda = 1$$
$$[A - 1I|0] = \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$E_1 = span\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

The matrix is not diagonalizable as long as $k \neq 0$ since the geometric multiplicity of $\lambda = 1$ will be less than its algebraic multiplicity. This matrix is only diagonalizable for $k = 0$.

Exercise 27

Find all real values of k for which A is diagonalizable.

$$A = \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$|A - \lambda I| = (1 - \lambda)^3 = 0$$
$$\lambda = 1$$
$$[A - 1I|0] = \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$E_1 = span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

By the same logic as Exercise 25, this matrix is only diagonalizable if $k = 0$.

Exercise 29

Find all real values of k for which A is diagonalizable.

$$A = \begin{bmatrix} 1 & 1 & k \\ 1 & 1 & k \\ 1 & 1 & k \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & k \\ 1 & 1-\lambda & k \\ 1 & 1 & k-\lambda \end{vmatrix} =$$

$$= (k - k + k\lambda) - (k - k\lambda - k) + (k - \lambda)((1 - \lambda)^2 - 1)$$

$$= k\lambda + k\lambda + (k - \lambda)(\lambda^2 - 2\lambda)$$

$$= 2k\lambda + k\lambda^2 - 2k\lambda - \lambda^3 + 2\lambda^2$$

$$= -\lambda^3 + (2 + k)\lambda^2$$

$$= \lambda^2(-\lambda + 2 + k) = 0$$

$$\lambda = 0 \quad \lambda = k + 2$$

$$[A - 0I|0] = \begin{bmatrix} 1 & 1 & k & 0 \\ 1 & 1 & k & 0 \\ 1 & 1 & k & 0 \end{bmatrix}$$

$$E_0 = span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} k \\ 0 \\ -1 \end{bmatrix}\right)$$

$$[A - (k+2)I|0] = \begin{bmatrix} -k-1 & 1 & k & 0 \\ 1 & -k-1 & k & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 2 & 0 \\ 1 & -k-1 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A is diagonalizable for all real values of k.

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech