

Linear Algebra: Homework 7

Alvin Lin

August 2016 - December 2016

Section 3.5

Exercise 1

Let S be the collection of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in \mathbb{R}^2 that satisfy the given property. In each case, either prove that S forms a subspace of \mathbb{R}^2 or give a counterexample to show that it does not.

$$x = 0 \\ S = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \text{span} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Since $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2$, S is a subspace of \mathbb{R}^2 .

Exercise 3

Let S be the collection of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in \mathbb{R}^2 that satisfy the given property. In each case, either prove that S forms a subspace of \mathbb{R}^2 or give a counterexample to show that it does not.

$$y = 2x \\ S = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \text{span} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$

Since $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$, S is a subspace of \mathbb{R}^2 .

Exercise 5

Let S be the collection of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in \mathbb{R}^3 that satisfy the given property. In each case, either prove that S forms a subspace of \mathbb{R}^3 or give a counterexample to show that it does not.

$$x = y = z \\ S = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

Since $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$, S is a subspace of \mathbb{R}^3 .

Exercise 7

Let S be the collection of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in \mathbb{R}^3 that satisfy the given property. In each case, either prove that S forms a subspace of \mathbb{R}^3 or give a counterexample to show that it does not.

$$x - y + z = 1$$

$$0 - 0 + 0 \neq 1$$

Since the zero vector is not in S , S is not a subspace of \mathbb{R}^3 .

Exercise 9

Prove that every line through the origin in \mathbb{R}^3 is a subspace of \mathbb{R}^3 .

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \text{span} \left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right)$$

Every line through the origin can be described as the span of a vector $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3$. Therefore, every line through the origin is a subspace of \mathbb{R}^3 .

Exercise 10

Suppose S consists of all points in \mathbb{R}^2 that are on the x-axis or the y-axis (or both). Is S a subspace of \mathbb{R}^2 ? Why or why not?

$$S = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \mid y \in \mathbb{R} \right\}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in S$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in S$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin S$$

The subspace S is not closed under addition.

Exercise 15

If A is the matrix in Exercise 11, is $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$ in $\text{null}(A)$?

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore \vec{v} \notin \text{null}(A)$$

Exercise 16

If A is the matrix in Exercise 12, is $\vec{v} = \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix}$ in $\text{null}(A)$?

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \vec{v} \in \text{null}(A)$

Exercise 17

Give bases for $\text{row}(A)$, $\text{col}(A)$, and $\text{null}(A)$.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

basis for $\text{row}(A) = \{[1 \ 0 \ -1], [0 \ 1 \ 2]\}$

basis for $\text{col}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$

$$1x_1 - 1x_3 = 0$$

$$1x_2 + 2x_3 = 0$$

$$\text{null}(A) = \begin{bmatrix} x_1 \\ -2x_1 \\ x_1 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

basis for $\text{null}(A) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

Exercise 19

Give bases for $\text{row}(A)$, $\text{col}(A)$, and $\text{null}(A)$.

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{basis for } \text{row}(A) = \left\{ \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \right\}$$

$$\text{basis for } \text{col}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$$A\vec{x} = 0$$

$$x_1 + x_3 = 0$$

$$x_2 - x_3 = 0$$

$$x_4 = 0$$

$$\text{null}(A) = \begin{bmatrix} x_1 \\ -x_1 \\ -x_1 \\ 0 \end{bmatrix}$$

$$\text{basis for } \text{null}(A) = \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Exercise 27

Find a basis for the span of the given vectors.

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ \text{basis} &= \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

Exercise 29

Find a basis for the span of the given vectors.

$$[2 \quad -3 \quad -1] \quad [1 \quad -1 \quad 0] \quad [4 \quad -4 \quad 1]$$

$$\begin{aligned} \begin{bmatrix} 2 & -3 & -1 \\ 1 & -1 & 0 \\ 4 & -4 & 1 \end{bmatrix} &= \begin{bmatrix} 0 & -1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \text{basis} &= \{e_1, e_2, e_3\} \end{aligned}$$

Exercise 35

Give the rank and nullity of the matrix in Exercise 17.

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \\ \text{rank}(A) &= 2 \\ \text{nullity}(A) &= 3 - \text{rank}(A) = 1 \end{aligned}$$

Exercise 37

Give the rank and nullity of the matrix in Exercise 19.

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \text{rank}(A) &= 3 \\ \text{nullity}(A) &= 4 - \text{rank}(A) = 1 \end{aligned}$$

Exercise 39

If A is a 3×5 matrix, explain why the columns of A must be linearly dependent.

$$\begin{aligned} \text{rank}(A) &\leq \min(m, n) \\ A\vec{x} &= \vec{0} \\ 5 &= \text{rank}(A) + \text{nullity}(A) \\ \text{rank}(A) &\leq 3 \end{aligned}$$

The nullity of A must be at least 2, therefore $A\vec{x} = \vec{0}$ has a non-trivial solution. Thus, the columns of A are linearly dependent. Since there are 5 columns and only 3 can be linearly independent, they must be linearly dependent.

Exercise 40

If A is a 4×2 matrix, explain why the rows of A must be linearly dependent.

There are four rows in total, but only two rows can be linearly independent in a 4×2 matrix, therefore the rows must be linearly dependent.

Exercise 41

If A is a 3×5 matrix, what are the possible values of $\text{nullity}(A)$?

$$\begin{aligned} \text{rank}(A) &\leq 3 \\ n &= \text{rank}(A) + \text{nullity}(A) \\ 5 &= \text{rank}(A) + \text{nullity}(A) \\ 5 - \text{nullity}(A) &\leq 3 \\ -\text{nullity}(A) &\leq -2 \\ \text{nullity}(A) &\geq 2 \\ 2 &\leq \text{nullity}(A) \leq 5 \end{aligned}$$

Exercise 42

If A is a 4×2 matrix, what are the possible values of $\text{nullity}(A)$?

$$\begin{aligned} \text{rank}(A) &\leq 2 \\ n &= \text{rank}(A) + \text{nullity}(A) \\ 2 &= \text{rank}(A) + \text{nullity}(A) \\ 2 - \text{nullity}(A) &\leq 2 \\ -\text{nullity}(A) &\leq 0 \\ \text{nullity}(A) &\geq 0 \\ 0 &\leq \text{nullity}(A) \leq 2 \end{aligned}$$

Exercise 43

Find all possible values of $\text{rank}(A)$ as a varies.

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 1 & -\frac{2}{a} & \frac{1}{a} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 0 & -\frac{2}{a}-2 & \frac{1}{a}-a \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 0 & -2-2a & 1-a^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & a \\ 0 & 2a+2 & 1+a \\ 0 & 2a+2 & a^2-1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & a \\ 0 & 2a+2 & 1+a \\ 0 & 0 & a^2-a-2 \end{bmatrix} \end{aligned}$$

$$2a+2=0$$

$$a=-1$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$a^2-a-2=0$$

$$(a-2)(a+1)=0$$

$$a=1, -2$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & a \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = \begin{cases} 1 & \text{if } a = 1 \\ 2 & \text{if } a = -1 \vee a = 2 \\ 3 & \text{otherwise} \end{cases}$$

Exercise 44

Find all possible values of $\text{rank}(A)$ as a varies.

$$\begin{aligned}
 A &= \begin{bmatrix} a & 2 & -1 \\ 3 & 3 & -2 \\ -2 & -1 & a \end{bmatrix} \\
 &= \begin{bmatrix} 1 & \frac{2}{a} & -\frac{1}{a} \\ 1 & 2 & -2+a \\ -2 & -1 & a \end{bmatrix} \\
 &= \begin{bmatrix} 1 & \frac{2}{a} & -\frac{1}{a} \\ 1 & 2 & -2+a \\ 0 & 3 & a+2(-2+a) \end{bmatrix} \\
 &= \begin{bmatrix} 1 & \frac{2}{a} & -\frac{1}{a} \\ 0 & 2-\frac{2}{a} & -2+a+\frac{1}{a} \\ 0 & 3 & 3a-4 \end{bmatrix} \\
 &= \begin{bmatrix} a & 2 & -1 \\ 0 & 2a-2 & -2a+a^2+1 \\ 0 & 3 & 3a-4 \end{bmatrix} \\
 a &= 1 \\
 &\rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & -1 \end{bmatrix} \\
 \text{rank}(A) &= \begin{cases} 2 & \text{if } a = 1 \\ 3 & \text{otherwise} \end{cases}
 \end{aligned}$$

Exercise 45

Do $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ form a basis for \mathbb{R}^3 ?

$$\begin{aligned}
 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

Since the vectors are linearly independent and form the standard basis vectors, they form a basis for \mathbb{R}^3 .

Exercise 47

Do $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ for a basis for \mathbb{R}^4 ?

$$\begin{aligned}
 \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

Since the vectors form the standard basis vectors, they form a basis for \mathbb{R}^4 .

Exercise 51

Show that w is in $\text{span}(B)$ and find the coordinate vector $[w]_B$.

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\} \quad \vec{w} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$$

$$\begin{aligned}
 c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} &= \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix} \\
 \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 6 \\ 0 & -1 & 2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & -1 & 2 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \\
 [w]_B &= \begin{bmatrix} 3 \\ -2 \end{bmatrix}
 \end{aligned}$$

Exercise 52

Show that w is in $\text{span}(B)$ and find the coordinate vector $[w]_B$.

$$B = \left\{ \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} \right\} \quad \vec{w} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{aligned} c_1 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} &= \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \\ \begin{bmatrix} 3 & 5 & 1 \\ 1 & 1 & 3 \\ 4 & 6 & 4 \end{bmatrix} &= \begin{bmatrix} 0 & 2 & -8 \\ 1 & 1 & 3 \\ 2 & 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & -4 \\ 1 & 1 & 3 \\ 0 & 1 & -4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \\ [w]_B &= \begin{bmatrix} 7 \\ -4 \end{bmatrix} \end{aligned}$$

Exercise 57

If A is $m \times n$, prove that every vector in $\text{null}(A)$ is orthogonal to every vector in $\text{row}(A)$.

$$\begin{aligned} A\vec{x} &= \vec{0} \quad \forall \vec{x} \in \mathbb{R}^n \\ \vec{y} &= \sum_{i=1}^m c_i \vec{a}_i \quad \forall \vec{y} \in \text{row}(A) \\ &= [\text{col}_1(A) \quad \dots \quad \text{col}_m(A)] \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \\ &= A^T \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \\ x^T y &= x^T A^T c \\ x^T y &= (Ax)^T c \\ x \cdot y &= 0c = 0 \end{aligned}$$

Exercise 58

If A and B are $n \times n$ of rank n , prove that AB has rank n .

$$\begin{aligned}
 \text{nullity}(A) &= 0 \\
 \text{nullity}(B) &= 0 \\
 B\vec{x} &= 0 \\
 A(B\vec{x}) &= 0 \\
 (AB)\vec{x} &= 0 \\
 \text{nullity}(AB) &= 0 \\
 \text{rank}(AB) &= n - \text{nullity}(AB) = n
 \end{aligned}$$

Exercise 59a

Prove that $\text{rank}(AB) \leq \text{rank}(B)$.

$$\begin{aligned}
 n &= \text{rank}(AB) + \text{nullity}(AB) \\
 &= \text{rank}(B) + \text{nullity}(B) \\
 \text{nullity}(B) &= \dim(\text{null}(B)) \\
 &\leq \dim(\text{null}(AB)) \\
 &\leq \text{nullity}(AB) \\
 \text{rank}(AB) + \text{nullity}(AB) &= \text{rank}(B) + \text{nullity}(B) \\
 \therefore \text{rank}(AB) &\leq \text{rank}(B)
 \end{aligned}$$

Exercise 60a

Prove that $\text{rank}(AB) \leq \text{rank}(A)$.

$$\begin{aligned}
 AB &= A [\text{col}_1(B) \ \dots \ \text{col}_n(B)] \\
 \text{col}(AB) &\subseteq \text{col}(B) \\
 \dim(\text{col}(AB)) &\leq \dim(\text{col}(B)) \\
 \text{rank}(AB) &\leq \text{rank}(B)
 \end{aligned}$$

Exercise 61

Prove that if U is invertible, then $\text{rank}(UA) = \text{rank}(A)$.

$$\begin{aligned}
 A &= IA \\
 A &= U^{-1}UA \\
 \text{rank}(U) &= n \\
 \therefore \text{rank}(A) &= n = \text{rank}(U)
 \end{aligned}$$

Prove that if V is invertible, then $\text{rank}(AV) = \text{rank}(A)$.

$$\begin{aligned}
 A &= AV^{-1}V \\
 \text{rank}(V) &= n \\
 \therefore \text{rank}(A) &= n = \text{rank}(V)
 \end{aligned}$$

Exercise 62

Prove that an $m \times n$ matrix A has rank 1 if and only if A can be written as the outer product uv^T of a vector $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$.

Suppose:

$$\begin{aligned}
 A &= \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} w \\
 u &= \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\
 w &= v^T \\
 A &= uv^T \\
 \text{rank}(A) &= \text{rank}(uv^T) \\
 &\leq \text{rank}(u) \\
 &\leq 1 \\
 \text{rank}(A) &= 1
 \end{aligned}$$

Exercise 63

Prove that an $m \times n$ matrix A has rank r , prove that A can be written as the sum of r matrices, each of which has rank 1.

$$\begin{aligned}
 \text{rank}(A) &= r \\
 &= \dim(\text{row}(A)) \\
 A &= \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \\
 &= \begin{bmatrix} c_{11}v_1 + c_{12}v_2 + \cdots + c_{1r}v_r \\ c_{21}v_1 + c_{22}v_2 + \cdots + c_{2r}v_r \\ \vdots \\ c_{m1}v_1 + c_{m2}v_2 + \cdots + c_{mr}v_r \end{bmatrix} \\
 &= \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix} v_1 + \cdots + \begin{bmatrix} c_{1r} \\ c_{2r} \\ \vdots \\ c_{mr} \end{bmatrix} v_r
 \end{aligned}$$

Because $\begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix}$, has rank 1, each of the r matrices has rank 1.

Exercise 64

Prove that, for $m \times n$ matrices A and B , $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.

$$\text{row}_i(A + B) = \text{row}_i(A) + \text{row}_i(B)$$

The rows of $A + B$ can be expressed as linear combinations of the respective rows of A and B .

Exercise 65

Let A be an $n \times n$ matrix such that $A^2 = 0$. Prove that $\text{rank}(A) \leq \frac{n}{2}$.

$$\begin{aligned} A^2 &= A [\text{col}_1(A) \ \dots \ \text{col}_n(A)] \\ &= 0 \\ A\vec{x} &= 0 \\ \text{col}(A) &\subseteq \text{null}(A) \\ \text{rank}(A) + \text{nullity}(A) &= n \\ \text{rank}(A) + \text{rank}(A) &\leq n \\ 2\text{rank}(A) &\leq n \\ \text{rank}(A) &\leq \frac{n}{2} \end{aligned}$$

Exercise 66

Let A be a skew-symmetric $n \times n$ matrix.

- Prove that $x^T Ax = 0$ for all $x \in \mathbb{R}^n$.

$$\begin{aligned} x^T Ax &= (x^T Ax)^T \\ &= (Ax)^T (x^T)^T \\ &= x^T A^T x \\ &= x^T (-A)x \\ x^T Ax &= -x^T Ax \\ \therefore x^T Ax &= 0 \end{aligned}$$

- Prove that $I + A$ is invertible.

If this is true $(I + A)x = 0$ has only the trivial solution.

$$\begin{aligned} (I + A)x &= 0 \\ x + Ax &= 0 \\ x^T x + x^T Ax &= 0(x^T) \\ x^T x + 0 &= 0 \\ \therefore x &= 0 \end{aligned}$$

Section 3.6

Exercise 1

Let $T_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the matrix transformation corresponding to $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$. Find $T_A(\vec{u})$ and $T_A(\vec{v})$, where $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

$$T_A(\vec{u}) = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$$

$$T_A(\vec{v}) = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

Exercise 3

Prove that the given transformation is a linear transformation, using the definition.

$$\begin{aligned} T \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x + y \\ x - y \end{bmatrix} \\ T(u + v) &= T \left(\begin{bmatrix} u_x + v_x \\ u_y + v_y \end{bmatrix} \right) \\ &= \begin{bmatrix} u_x + v_x + u_y + v_y \\ u_x + v_x - u_y - v_y \end{bmatrix} \\ &= \begin{bmatrix} (u_x + u_y) + (v_x + v_y) \\ (u_x - u_y) + (v_x - v_y) \end{bmatrix} \\ &= T(u) + T(v) \\ T(cu) &= \begin{bmatrix} cu_x + cu_y \\ cu_x - cu_y \end{bmatrix} \\ &= c \begin{bmatrix} u_x + u_y \\ u_x - u_y \end{bmatrix} \\ &= cT(u) \end{aligned}$$

Exercise 5

Prove that the given transformation is a linear transformation, using the definition.

$$\begin{aligned}
 T \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} x - y + z \\ 2x + y - 3z \end{bmatrix} \\
 T(u + v) &= T \left(\begin{bmatrix} u_x + v_x \\ u_y + v_y \end{bmatrix} \right) \\
 &= \begin{bmatrix} (u_x + v_x) - (u_y + v_y) + (u_z + v_z) \\ 2(u_x + v_x) + (u_y + v_y) - 3(u_z + v_z) \end{bmatrix} \\
 &= \begin{bmatrix} (u_x - u_y + u_z) + (v_x - v_y + v_z) \\ (2u_x + u_y - 3u_z) + (2v_x + v_y - 3v_z) \end{bmatrix} \\
 &= T(u) + T(v) \\
 T(cu) &= \begin{bmatrix} cu_x - cu_y + cu_z \\ c2u_x + cu_y - c3u_z \end{bmatrix} \\
 &= c \begin{bmatrix} u_x - u_y + u_z \\ 2u_x + u_y - 3u_z \end{bmatrix} \\
 &= cT(u)
 \end{aligned}$$

Exercise 7

Give a counterexample to show that the given transformation is not a linear transformation.

$$\begin{aligned}
 T \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} y \\ x^2 \end{bmatrix} \\
 T \left(2 \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right) &= \begin{bmatrix} 6 \\ 36 \end{bmatrix} \\
 &\neq 2T \left(\begin{bmatrix} 3 \\ 3 \end{bmatrix} \right) \\
 &\neq \begin{bmatrix} 6 \\ 18 \end{bmatrix}
 \end{aligned}$$

Exercise 9

Give a counterexample to show that the given transformation is not a linear transformation.

$$\begin{aligned}
 T \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} xy \\ x + y \end{bmatrix} \\
 T \left(2 \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right) &= \begin{bmatrix} 36 \\ 12 \end{bmatrix} \\
 &\neq 2T \left(\begin{bmatrix} 3 \\ 3 \end{bmatrix} \right) \\
 &\neq \begin{bmatrix} 18 \\ 12 \end{bmatrix}
 \end{aligned}$$

Exercise 11

Find the standard matrix of the linear transformation.

$$\begin{aligned} T \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x + y \\ x - y \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

Exercise 13

Find the standard matrix of the linear transformation.

$$\begin{aligned} T \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} x - y + z \\ 2x + y - 3z \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix} \end{aligned}$$

Exercise 40

Use matrices to prove the given statements about the transformations from \mathbb{R}^2 to \mathbb{R}^2 . If R_θ denotes a rotation (about the origin) through the angle θ , then $R_\alpha \circ R_\beta = R_{\alpha + \beta}$.

$$\begin{aligned} R_\alpha &= \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \\ R_\beta &= \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \\ R_\alpha \circ R_\beta &= \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \\ &= \begin{bmatrix} \cos \alpha \cos \beta - \sin \alpha \sin \beta & -\cos \alpha \sin \beta - \sin \alpha \cos \beta \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta & -\sin \alpha \sin \beta + \cos \alpha \cos \beta \end{bmatrix} \\ &= \begin{bmatrix} \cos \alpha \cos \beta - \sin \alpha \sin \beta & -(\cos \alpha \sin \beta + \sin \alpha \cos \beta) \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta & \cos \alpha \cos \beta - \sin \alpha \sin \beta \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix} \\ &= R_{\alpha + \beta} \end{aligned}$$

Exercise 42

(a) If P is a projection, then $P \circ P = P$.

$$\begin{aligned} P &= \begin{bmatrix} \frac{d_x^2}{d_x^2 + d_y^2} & \frac{d_x d_y}{d_x^2 + d_y^2} \\ \frac{d_x d_y}{d_x^2 + d_y^2} & \frac{d_y^2}{d_x^2 + d_y^2} \end{bmatrix} \\ P \circ P &= \begin{bmatrix} \frac{d_x^4 + d_x^2 d_y^2}{(d_x^2 + d_y^2)^2} & \frac{d_x^3 d_y + d_x d_y^3}{(d_x^2 + d_y^2)^2} \\ \frac{d_x^3 d_y + d_x d_y^3}{(d_x^2 + d_y^2)^2} & \frac{d_y^4 + d_x^2 d_y^2}{(d_x^2 + d_y^2)^2} \end{bmatrix} \\ &= P \end{aligned}$$

Exercise 44

Let T be a linear transformation from \mathbb{R}^2 to \mathbb{R}^2 . Prove that T maps a straight line to a straight line or a point.

$$\begin{aligned} l &= \vec{x} + t\vec{d} \\ T(\vec{x} + t\vec{d}) &= T(\vec{x}) + T(t\vec{d}) \\ &= T(\vec{x}) + tT(\vec{d}) \end{aligned}$$

When $t = 0$, the result is a point, otherwise, the resulting mapping is a line.

Exercise 52

Prove that $P_l(c\vec{v}) = cP_l(\vec{v})$ for an scalar c .

$$\begin{aligned} P_l(c\vec{v}) &= \left(\frac{\vec{d} \cdot (c\vec{v})}{\vec{d} \cdot \vec{d}} \vec{d} \right) \\ &= \left(\frac{c(\vec{d} \cdot \vec{v})}{\vec{d} \cdot \vec{d}} \vec{d} \right) \\ &= c \left(\frac{\vec{d} \cdot \vec{v}}{\vec{d} \cdot \vec{d}} \vec{d} \right) \\ &= cP_l(\vec{v}) \end{aligned}$$

Exercise 53

Prove that $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation if and only if:

$$T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2)$$

$$\begin{aligned} T(c_1\vec{v}_1 + c_2\vec{v}_2) &= T(c_1\vec{v}_1) + T(c_2\vec{v}_2) \\ &= c_1T(\vec{v}_1) + c_2T(\vec{v}_2) \end{aligned}$$

Exercise 54

Prove that (as noted at the beginning of this section) the range of a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the column space of its matrix $[T]$.

$$\begin{aligned} [T] &= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \\ \vec{u} &= \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \\ T(\vec{u}) &= u_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + u_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \end{aligned}$$

The range of T is a linear combination of the columns, so it is a subset of the column space. The converse is also true, the column space is a subset of the range. Therefore, the two sets are equal.

If you have any questions, comments, or concerns, please contact me at alvin@omgimanagerd.tech