Linear Algebra: Homework 7

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Section 3.5

Exercise 1

Let S be the collection of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ that satisfy the given property. In each case, either prove that S forms a subspace of $\mathbb{R}^2$ or give a counterexample to show that it does not.

$$x = 0$$

$$S = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = span\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Since $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2$, S is a subspace of $\mathbb{R}^2$.

Exercise 3

Let S be the collection of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ that satisfy the given property. In each case, either prove that S forms a subspace of $\mathbb{R}^2$ or give a counterexample to show that it does not.

$$y = 2x$$

$$S = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} = span\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$

Since $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$, S is a subspace of $\mathbb{R}^2$.

Exercise 5

Let S be the collection of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ that satisfy the given property. In each case, either prove that S forms a subspace of $\mathbb{R}^2$ or give a counterexample to show that it does not.

$$x = y = z$$

$$S = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

Since $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$, S is a subspace of $\mathbb{R}^3$.

Exercise 7

Let S be the collection of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ that satisfy the given property. In each case, either prove that S forms a subspace of $\mathbb{R}^2$ or give a counterexample to show that it does not.

$$x - y + z = 1$$
$$0 - 0 + 0 \neq 1$$

Since the zero vector is not in S, S is not a subspace of $\mathbb{R}^3$.

Exercise 9

Prove that every line through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = span \left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right)$$

Every line through the origin can be described as the span of a vector $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3$. Therefore, every line through the origin is a subspace of $\mathbb{R}^3$.

Exercise 10

Suppose S consists of all points in $\mathbb{R}^2$ that are on the x-axis or the y-axis (or both). Is S a subspace of $\mathbb{R}^2$? Why or why not?

$$S = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \mid y \in \mathbb{R} \right\}$$
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in S$$
$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in S$$
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin S$$

The subspace S is not closed under addition.

Exercise 15

If A is the matrix in Exercise 11, is $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$ in $null(A)$?

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$
$$A\vec{v} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\therefore \vec{v} \notin null(A)$$

Exercise 16

If A is the matrix in Exercise 12, is $\vec{v} = \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix}$ in $null(A)$?

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \vec{v} \in null(A)$$

Exercise 17

Give bases for $row(A), col(A),$ and $null(A)$.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\text{basis for } row(A) = \left\{ \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \right\}$$

$$\text{basis for } col(A) = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$

$$1x_1 - 1x_3 = 0$$

$$1x_2 + 2x_3 = 0$$

$$null(A) = \begin{bmatrix} x_1 \\ -2x_1 \\ x_1 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{basis for } null(A) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Exercise 19

Give bases for $row(A)$, $col(A)$, and $null(A)$.

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{basis for } row(A) = \left\{ \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \right\}$$

$$\text{basis for } col(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$$A\vec{x} = 0$$

$$x_1 + x_3 = 0$$

$$x_2 - x_3 = 0$$

$$x_4 = 0$$

$$null(A) = \begin{bmatrix} x_1 \\ -x_1 \\ -x_1 \\ 0 \end{bmatrix}$$

$$\text{basis for } null(A) = \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Exercise 27

Find a basis for the span of the given vectors.

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{basis} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Exercise 29

Find a basis for the span of the given vectors.

$$\begin{bmatrix} 2 & -3 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 4 & -4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & -1 \\ 1 & -1 & 0 \\ 4 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{basis} = \{e_1, e_2, e_3\}$$

Exercise 35

Give the rank and nullity of the matrix in Exercise 17.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$rank(A) = 2$$

$$nullity(A) = 3 - rank(A) = 1$$

Exercise 37

Give the rank and nullity of the matrix in Exercise 19.

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$rank(A) = 3$$

$$nullity(A) = 4 - rank(A) = 1$$

Exercise 39

If A is a 3×5 matrix, explain why the columns of A must be linearly dependent.

$$rank(A) \leq min(m, n)$$
$$A\vec{x} = \vec{0}$$
$$5 = rank(A) + nullity(A)$$
$$rank(A) \leq 3$$

The nullity of A must be at least 2, therefore $A\vec{x} = \vec{0}$ has a non-trivial solution. Thus, the columns of A are linearly dependent. Since there are 5 columns and only 3 can be linearly independent, they must be linearly dependent.

Exercise 40

If A is a 4×2 matrix, explain why the rows of A must be linearly dependent.
There are four rows in total, but only two rows can be linearly independent in a 4×2 matrix, therefore the rows must be linearly dependent.

Exercise 41

If A is a 3×5 matrix, what are the possible values of $nullity(A)$?

$$rank(A) \leq 3$$
$$n = rank(A) + nullity(A)$$
$$5 = rank(A) + nullity(A)$$
$$5 - nullity(A) \leq 3$$
$$-nullity(A) \leq -2$$
$$nullity(A) \geq 2$$
$$2 \leq nullity(A) \leq 5$$

Exercise 42

If A is a 4×2 matrix, what are the possible values of $nullity(A)$?

$$rank(A) \leq 2$$
$$n = rank(A) + nullity(A)$$
$$2 = rank(A) + nullity(A)$$
$$2 - nullity(A) \leq 2$$
$$-nullity(A) \leq 0$$
$$nullity(A) \geq 0$$
$$0 \leq nullity(A) \leq 2$$

Exercise 43

Find all possible values of $rank(A)$ as a varies.

$$A = \begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 1 & -\frac{2}{a} & \frac{1}{a} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 0 & -\frac{2}{a}-2 & \frac{1}{a}-a \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 0 & -2-2a & 1-a^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & a \\ 0 & 2a+2 & 1+a \\ 0 & 2a+2 & a^2-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & a \\ 0 & 2a+2 & 1+a \\ 0 & 0 & a^2-a-2 \end{bmatrix}$$

$$2a + 2 = 0$$
$$a = -1$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$a^2 - a - 2 = 0$$
$$(a-2)(a+1) = 0$$
$$a = 1, -2$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & a \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$rank(A) = \begin{cases} 1 & \text{if } a = 1 \\ 2 & \text{if } a = -1 \lor a = 2 \\ 3 & \text{otherwise} \end{cases}$$

Exercise 44

Find all possible values of $rank(A)$ as a varies.

$$A = \begin{bmatrix} a & 2 & -1 \\ 3 & 3 & -2 \\ -2 & -1 & a \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{2}{a} & -\frac{1}{a} \\ 1 & 2 & -2+a \\ -2 & -1 & a \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{2}{a} & -\frac{1}{a} \\ 1 & 2 & -2+a \\ 0 & 3 & a+2(-2+a) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{2}{a} & -\frac{1}{a} \\ 0 & 2-\frac{2}{a} & -2+a+\frac{1}{a} \\ 0 & 3 & 3a-4 \end{bmatrix}$$

$$= \begin{bmatrix} a & 2 & -1 \\ 0 & 2a-2 & -2a+a^2+1 \\ 0 & 3 & 3a-4 \end{bmatrix}$$

$$a = 1$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & -1 \end{bmatrix}$$

$$rank(A) = \begin{cases} 2 & \text{if } a = 1 \\ 3 & \text{otherwise} \end{cases}$$

Exercise 45

Do $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{R}^3$?

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the vectors are linearly independent and form the standard basis vectors, they form a basis for $\mathbb{R}^3$.

Exercise 47

Do $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ for a basis for $\mathbb{R}^4$?

$$
\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Since the vectors form the standard basis vectors, they form a basis for $\mathbb{R}^4$.

Exercise 51

Show that w is in $span(B)$ and find the coordinate vector $[w]_B$.

$$
B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\} \quad \vec{w} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}
$$

$$
c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 6 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & -1 & 2 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
[w]_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}
$$

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Exercise 52

Show that w is in $span(B)$ and find the coordinate vector $[w]_B$.

$$B = \left\{ \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} \right\} \quad \vec{w} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 & 1 \\ 1 & 1 & 3 \\ 4 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -8 \\ 1 & 1 & 3 \\ 2 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -4 \\ 1 & 1 & 3 \\ 0 & 1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[w]_B = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

Exercise 57

If A is $m \times n$, prove that every vector in $null(A)$ is orthogonal to every vector in $row(A)$.

$$A\vec{x} = \vec{0} \quad \forall \vec{x} \in \mathbb{R}^m$$

$$\vec{y} = \sum_{i=1}^{m} c_i a_i \quad \forall \vec{u} \in row(A)$$

$$= \begin{bmatrix} col_1(A) & \cdots & col_m(A) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

$$= A^T \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

$$x^T y = x^T A^T c$$
$$x^T y = (Ax)^T c$$
$$x \cdot y = 0c = 0$$

Exercise 58

If A and B are $n \times n$ of rank n, prove that AB has rank n.

$$nullity(A) = 0$$
$$nullity(B) = 0$$
$$B\vec{x} = 0$$
$$A(B\vec{x}) = 0$$
$$(AB)\vec{x} = 0$$
$$nullity(AB) = 0$$
$$rank(AB) = n - nullity(AB) = n$$

Exercise 59a

Prove that $rank(AB) \leq rank(B)$.

$$n = rank(AB) + nullity(AB)$$
$$= rank(B) + nullity(B)$$
$$nullity(B) = dim(null(B))$$
$$\leq dim(null(AB))$$
$$\leq nullity(AB)$$
$$rank(AB) + nullity(AB) = rank(B) + nullity(B)$$
$$\therefore rank(AB) \leq rank(B)$$

Exercise 60a

Prove that $rank(AB) \leq rank(A)$.

$$AB = A \begin{bmatrix} col_1(B) & \cdots & col_n(B) \end{bmatrix}$$
$$col(AB) \subseteq col(B)$$
$$dim(col(AB)) \leq dim(col(B))$$
$$rank(AB) \leq rank(B)$$

Exercise 61

Prove that if U is invertible, then $rank(UA) = rank(A)$.

$$A = IA$$
$$A = U^{-1}UA$$
$$rank(U) = n$$
$$\therefore rank(A) = n = rank(U)$$

Prove that if V is invertible, then $rank(AV) = rank(A)$.

$$A = AI$$
$$A = AV^{-1}V$$
$$rank(U) = n$$
$$\therefore rank(A) = n = rank(V)$$

Exercise 62

Prove that an $m \times n$ matrix A has rank 1 if and only if A can be written as the outer product uv^T of a vector $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$.

Suppose:

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} w$$

$$u = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$w = v^T$$

$$A = uv^T$$

$$rank(A) = rank(uv^T)$$

$$\leq rank(u)$$

$$\leq 1$$

$$rank(A) = 1$$

Exercise 63

Prove that an $m \times n$ matrix A has rank r, prove that A can be written as the sum of r matrices, each of which has rank 1.

$$rank(A) = r$$

$$= dim(row(A))$$

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$$

$$= \begin{bmatrix} c_{11}v_1 + c_{12}v_2 + \cdots + c_{1r}v_r \\ c_{21}v_1 + c_{22}v_2 + \cdots + c_{2r}v_r \\ \vdots \\ c_{m1}v_1 + v_{m2}v_2 + \cdots + c_{mr}v_r \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix} v_1 + \cdots + \begin{bmatrix} c_{1r} \\ c_{2r} \\ \vdots \\ c_{mr} \end{bmatrix} v_r$$

Because $\begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix}$, has rank 1, each of the r matrices has rank 1.

Exercise 64

Prove that, for $m \times n$ matrices A and B, $rank(A + B) \leq rank(A) + rank(B)$.

$$row_i(A + B) = row_i(A) + row_i(B)$$

The rows of $A + B$ can be expressed as linear combinations of the respective rows of A and B.

Exercise 65

Let A be an $n \times n$ matrix such that $A^2 = 0$. Prove that $rank(A) \leq \frac{n}{2}$.

$$
\begin{aligned}
A^2 &= A \begin{bmatrix} col_1(A) & \cdots & col_n(A) \end{bmatrix} \\
&= 0 \\
A\vec{x} &= 0 \\
col(A) &\subseteq null(A) \\
rank(A) + nullity(A) &= n \\
rank(A) + rank(A) &\leq n \\
2rank(A) &\leq n \\
rank(A) &\leq \frac{n}{2}
\end{aligned}
$$

Exercise 66

Let A be a skew-symmetric $n \times n$ matrix.

- Prove that $x^T A x = 0$ for all $x \in \mathbb{R}^n$.

$$
\begin{aligned}
x^T A x &= (x^T A x)^T \\
&= (Ax)^T (x^T)^T \\
&= x^T A^T x \\
&= x^T (-A) x \\
x^T A x &= -x^T A x \\
\therefore x^T A x &= 0
\end{aligned}
$$

- Prove that $I + A$ is invertible.
  If this is true $(I + A)x = 0$ has only the trivial solution.

$$
\begin{aligned}
(I + A)x &= 0 \\
x + Ax &= 0 \\
x^T x + x^T A x &= 0(x^T) \\
x^T x + 0 &= 0 \\
\therefore x &= 0
\end{aligned}
$$

Section 3.6

Exercise 1

Let $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix transformation corresponding to $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$. Find $T_A(\vec{u})$ and $T_A(\vec{v})$, where $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

$$T_A(\vec{u}) = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$$

$$T_A(\vec{v}) = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

Exercise 3

Prove that the given transformation is a linear transformation, using the definition.

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ x - y \end{bmatrix}$$

$$T(u + v) = T \left(\begin{bmatrix} u_x + v_x \\ u_y + v_y \end{bmatrix} \right)$$

$$= \begin{bmatrix} u_x + v_x + u_y + v_y \\ u_x + v_x - u_y - v_y \end{bmatrix}$$

$$= \begin{bmatrix} (u_x + u_y) + (v_x + v_y) \\ (u_x - u_y) + (v_x - v_y) \end{bmatrix}$$

$$= T(u) + T(v)$$

$$T(cu) = \begin{bmatrix} cu_x + cu_y \\ cu_x - cu_y \end{bmatrix}$$

$$= c \begin{bmatrix} u_x + u_y \\ u_x - u_y \end{bmatrix}$$

$$= cT(u)$$

Exercise 5

Prove that the given transformation is a linear transformation, using the definition.

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - y + z \\ 2x + y - 3z \end{bmatrix}$$

$$T(u + v) = T\left(\begin{bmatrix} u_x + v_x \\ u_y + v_y \end{bmatrix}\right)$$

$$= \begin{bmatrix} (u_x + v_x) - (u_y + v_y) + (u_z + v_z) \\ 2(u_x + v_x) + (u_y + v_y) - 3(u_z + v_z) \end{bmatrix}$$

$$= \begin{bmatrix} (u_x - u_y + u_z) + (v_x - v_y + v_z) \\ (2u_x + u_y - 3u_z) + (2v_x + v_y - 3v_z) \end{bmatrix}$$

$$= T(u) + T(v)$$

$$T(cu) = \begin{bmatrix} cu_x - cu_y + cu_z \\ c2u_x + cu_y - c3u_z \end{bmatrix}$$

$$= c\begin{bmatrix} u_x - u_y + u_z \\ 2u_x + u_y - 3u_z \end{bmatrix}$$

$$= cT(u)$$

Exercise 7

Give a counterexample to show that the given transformation is not a linear transformation.

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x^2 \end{bmatrix}$$

$$T\left(2\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 36 \end{bmatrix}$$

$$\neq 2T\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right)$$

$$\neq \begin{bmatrix} 6 \\ 18 \end{bmatrix}$$

Exercise 9

Give a counterexample to show that the given transformation is not a linear transformation.

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ x + y \end{bmatrix}$$

$$T\left(2\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 36 \\ 12 \end{bmatrix}$$

$$\neq 2T\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right)$$

$$\neq \begin{bmatrix} 18 \\ 12 \end{bmatrix}$$

Exercise 11

Find the standard matrix of the linear transformation.

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ x - y \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Exercise 13

Find the standard matrix of the linear transformation.

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - y + z \\ 2x + y - 3z \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix}$$

Exercise 40

Use matrices to prove the given statements about the transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. If R_θ denotes a rotation (about the origin) through the angle θ, then $R_\alpha \circ R_\beta = R_\alpha + R_\beta$.

$$R_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$R_\beta = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$R_\alpha \circ R_\beta = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -(\cos\alpha\sin\beta + \sin\alpha\cos\beta) \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix}$$

$$= R_{\alpha+\beta}$$

Exercise 42

(a) If P is a projection, then $P \circ P = P$.

$$P = \begin{bmatrix} \frac{d_x^2}{d_x^2+d_y^2} & \frac{d_x d_y}{d_x^2+d_y^2} \\ \frac{d_x d_y}{d_x^2+d_y^2} & \frac{d_y^2}{d_x^2+d_y^2} \end{bmatrix}$$

$$P \circ P = \begin{bmatrix} \frac{d_x^4+d_x^2 d_y^2}{(d_x^2+d_y^2)^2} & \frac{d_x^3 d_y+d_x d_y^3}{(d_x^2+d_y^2)^2} \\ \frac{d_y^3 d_y+d_x d_y^3}{(d_x^2+d_y^2)^2} & \frac{d_y^4+d_x^2 d_y^2}{(d_1^2+d_2^2)^2} \end{bmatrix}$$

$$= P$$

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Exercise 44

Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Prove that T maps a straight line to a straight line or a point.

$$l = \vec{x} + t\vec{d}$$
$$T(\vec{x} + t\vec{d}) = T(\vec{x}) + T(t\vec{d})$$
$$= T(\vec{x}) + tT(\vec{d})$$

When $t = 0$, the result is a point, otherwise, the resulting mapping is a line.

Exercise 52

Prove that $P_l(c\vec{v}) = cP_l(\vec{v})$ for an scalar c.

$$P_l(c\vec{v}) = \left(\frac{\vec{d} \cdot (c\vec{v})}{\vec{d} \cdot \vec{d}} \vec{d} \right)$$
$$= \left(\frac{c(\vec{d} \cdot \vec{v})}{\vec{d} \cdot \vec{d}} \vec{d} \right)$$
$$= c \left(\frac{\vec{d} \cdot \vec{v}}{\vec{d} \cdot \vec{d}} \vec{d} \right)$$
$$= cP_l(\vec{v})$$

Exercise 53

Prove that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if:
$$T(c_1\vec{v_1} + c_2\vec{v_2}) = c_1T(\vec{v_1}) + c_2T(\vec{v_2})$$

$$T(c_1\vec{v_1} + c_2\vec{v_2}) = T(c_1\vec{v_1}) + T(c_2\vec{v_2})$$
$$= c_1T(\vec{v_1}) + c_2T(\vec{v_2})$$

Exercise 54

Prove that (as noted at the beginning of this section) the range of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the column space of its matrix $[T]$.

$$[T] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

$$T(\vec{u}) = u_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + u_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

The range of T is a linear combination of the columns, so it is a subset of the column space. The converse is also true, the column space is a subset of the range. Therefore, the two sets are equal.

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech