Linear Algebra: Homework 5

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Section 2.3

Exercise 1

Determine if the vector $\vec{v}$ is a linear combination of the remaining vectors.

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{u_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \vec{u_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 3 \\ -1 & -12 & \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 3 \\ -1 & 05 & \end{bmatrix} \\ &\text{Yes} \end{aligned}$$

Exercise 3

Determine if the vector $\vec{v}$ is a linear combination of the remaining vectors.

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{u_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{u_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix} \\ &\text{No} \end{aligned}$$

Exercise 5

Determine if the vector $\vec{v}$ is a linear combination of the remaining vectors.

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{u_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{u_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \vec{u_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 4 \\ 0 & 1 & 1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

Yes

Exercise 7

Determine if the vector $\vec{v}$ is in the span of the columns of the matrix A.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$
\begin{aligned}
A|\vec{b} &= \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 5 \\ 1 & 0 & -4 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2 & 9 \\ 1 & 0 & -4 \end{bmatrix}
\end{aligned}
$$

There is a unique solution for:

$$
\begin{aligned}
1a + 2b &= 5 \\
3a + 4b &= 6
\end{aligned}
$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{9}{2} \end{bmatrix}$$

Exercise 9

Show that $\mathbb{R}^2 = span\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right)$

$$\begin{aligned}
\begin{bmatrix}1 & 1 & a\\1 & -1 & b\end{bmatrix} &= \begin{bmatrix}1 & 1 & a\\2 & 0 & a+b\end{bmatrix}\\
&= \begin{bmatrix}1 & 1 & a\\1 & 0 & \frac{a+b}{2}\end{bmatrix}\\
&= \begin{bmatrix}0 & 1 & a-\frac{a+b}{2}\\1 & 0 & \frac{a+b}{2}\end{bmatrix}\\
x &= a - \frac{a+b}{2}\\
y &= \frac{a+b}{2}
\end{aligned}$$

Any point in $\mathbb{R}^2$ can be described by a linear combination of the two vectors using the constants described above.

Exercise 11

Show that $\mathbb{R}^3 = span\left(\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right)$

$$\begin{aligned}
\begin{bmatrix}1 & 1 & 0 & a\\0 & 1 & 1 & b\\1 & 0 & 1 & c\end{bmatrix} &= \begin{bmatrix}1 & 0 & -1 & a-b\\0 & 1 & 1 & b\\1 & 0 & 1 & c\end{bmatrix}\\
&= \begin{bmatrix}2 & 0 & 0 & a-b+c\\0 & 1 & 1 & b\\1 & 0 & 1 & c\end{bmatrix}\\
&= \begin{bmatrix}1 & 0 & 0 & \frac{a-b+c}{2}\\0 & 1 & 1 & b\\0 & 0 & 1 & c-\frac{a-b+c}{2}\end{bmatrix}\\
&= \begin{bmatrix}1 & 0 & 0 & \frac{a-b+c}{2}\\0 & 1 & 0 & b-(c-\frac{a-b+c}{2})\\0 & 0 & 1 & c-\frac{a-b+c}{2}\end{bmatrix}\\
x &= \frac{a-b+c}{2}\\
y &= b-(c-\frac{a-b+c}{2})\\
z &= c-\frac{a-b+c}{2}
\end{aligned}$$

Any point in $\mathbb{R}^3$ can be described by a linear combination of the two vectors using the constants described above.

Exercise 13b

Describe the span of the given vectors algebraically.

$$\begin{bmatrix}2\\-4\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}$$

$$
\begin{aligned}
s\begin{bmatrix}2\\-4\end{bmatrix}+t\begin{bmatrix}-1\\2\end{bmatrix}&=\begin{bmatrix}x\\y\end{bmatrix}\\
2s-t&=x\\
-4s+2t&=y\\
\begin{bmatrix}2&-1&x\\-4&2&y\end{bmatrix}&=\begin{bmatrix}2&-1&x\\0&0&y+2x\end{bmatrix}\\
y+2x&=0\\
y&=-2x
\end{aligned}
$$

The span of the two vectors describes the set of all vectors parallel and antiparallel to the given vectors, which line on the line $y=-2x$.

Exercise 15b

Describe the span of the given vectors algebraically.

$$
\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}3\\2\\-1\end{bmatrix}
$$

$$
\begin{aligned}
s\begin{bmatrix}1\\2\\0\end{bmatrix}+t\begin{bmatrix}3\\2\\-1\end{bmatrix}&=\begin{bmatrix}x\\y\\z\end{bmatrix}\\
s+3t&=x\\
2s+2t&=y\\
-t&=z\\
\begin{bmatrix}1&3&x\\2&2&y\\0&-1&z\end{bmatrix}&=\begin{bmatrix}1&0&x+3z\\2&0&y+2z\\0&-1&z\end{bmatrix}\\
&=\begin{bmatrix}1&0&x+3z\\0&0&y+2z-2(x+3z)\\0&-1&z\end{bmatrix}\\
0&=y+2z-2x-6z\\
&=-2x+y-4z
\end{aligned}
$$

The span of the vectors describes a plane with the equation $0=-2x+y-4z$.

Exercise 17

The general equation of the plane that contains the points (1,0,3),(-1,1,-3), and the origin is of the form $ax + by + cz = 0$. Solve for a, b, c.

$$\begin{aligned}
\begin{bmatrix} 1 & -1 & 0 & x \\ 0 & 1 & 0 & y \\ 3 & -3 & 0 & z \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & x+y \\ 0 & 1 & 0 & y \\ 1 & -1 & 0 & \frac{z}{3} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & x+y \\ 0 & 1 & 0 & y \\ 1 & 0 & 0 & \frac{z}{3}+y \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & x+y \\ 0 & 1 & 0 & y \\ 0 & 0 & 0 & \frac{z}{3}+y-(x+y) \end{bmatrix} \\
0 &= \frac{z}{3} + y - x - y \\
&= -x + \frac{z}{3} \\
&= -3x + z \\
a &= -3 \\
b &= 0 \\
c &= 1
\end{aligned}$$

Exercise 18

Prove that $\vec{u}, \vec{v}, \vec{w}$ are all in $span(\vec{u}, \vec{v}, \vec{w})$.

$$\begin{aligned}
\vec{u} &= 1\vec{u} + 0\vec{v} + 0\vec{w} \\
\vec{v} &= 0\vec{u} + 1\vec{v} + 0\vec{w} \\
\vec{w} &= 0\vec{u} + 0\vec{v} + 1\vec{w} \\
\vec{u}, \vec{v}, \vec{w} &\in span(\vec{u}, \vec{v}, \vec{w})
\end{aligned}$$

Exercise 19

$$\begin{aligned}
\vec{u} &= 1\vec{u} + 0(\vec{u}+\vec{v}) + 0(\vec{u}+\vec{v}+\vec{w}) \\
\vec{v} &= -1\vec{u} + 1(\vec{u}+\vec{v}) + 0(\vec{u}+\vec{v}+\vec{w}) \\
\vec{w} &= 0\vec{u} - 1(\vec{u}+\vec{v}) + 1(\vec{u}+\vec{v}+\vec{w}) \\
\vec{u}, \vec{v}, \vec{w} &\in span(\vec{u}, \vec{u}+\vec{v}, \vec{u}+\vec{v}+\vec{w})
\end{aligned}$$

Exercise 20

Prove that if $\vec{u_1}, \dots, \vec{u_m} \in \mathbb{R}^n$ where $S = \{\vec{u_1}, \dots, \vec{u_k}\}$ and $T = \{\vec{u_1}, \dots, \vec{u_k}, \vec{u_{k+1}}, \dots, \vec{u_m}\}$ that $span(S) \subseteq span(T)$.

$$\begin{aligned}
c_1\vec{u_1} + c_2\vec{u_2} + \cdots + c_k\vec{u_k} &\in span(S) \\
c_1\vec{u_1} + c_2\vec{u_2} + \cdots + c_k\vec{u_k} &= c_1\vec{u_1} + \cdots + c_k\vec{u_k} + 0\vec{u_{k+1}} + \cdots + 0\vec{u_m} \\
c_1\vec{u_1} + \cdots + c_k\vec{u_k} + 0\vec{u_{k+1}} + \cdots + 0\vec{u_m} &\in span(T) \\
span(S) &\subseteq span(T)
\end{aligned}$$

Deduce if $\mathbb{R}^n = span(S)$, then $\mathbb{R}^n = span(T)$.

$$\begin{aligned} \mathbb{R}^n &= span(S) \\ span(S) &\subseteq span(T) \\ span(T) &\supseteq \mathbb{R}^n \\ span(T) &= \mathbb{R}^n \end{aligned}$$

Exercise 21

Suppose $\vec{w}$ is a linear combination of $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_k}$ and each $\vec{u_1}$ is a linear combination of $\vec{v_1}, \ldots, \vec{v_m}$. Prove $\vec{w}$ is a linear combination of $\vec{v_1}, \ldots, \vec{v_m}$.

$$\begin{aligned} \vec{w} &= \sum_{i=1}^{k} c_i u_i \\ &= \sum_{i=1}^{k} c_i \left(\sum_{j=1}^{m} d_{ij} \vec{v_j} \right) \\ &= \sum_{i=1}^{k} \left(\sum_{j=1}^{m} c_i d_{ij} \vec{v_j} \right) \\ &= \sum_{j=1}^{m} \left(\sum_{i=1}^{k} c_i d_{ij} \right) \vec{v_i} \\ w &\in span(\vec{v_1}, \ldots, \vec{v_m}) \end{aligned}$$

Also suppose each $\vec{v_j}$ is a linear combination of $\vec{u_1}, \ldots, \vec{u_k}$. Prove $span(\vec{u_1}, \ldots, \vec{u_k}) = span(\vec{v_1}, \ldots, \vec{v_m})$. Above we proved:

$$span(\vec{u_1}, \ldots, \vec{u_k}) \subseteq span(\vec{v_1}, \ldots, \vec{v_m})$$

Therefore:

$$span(\vec{u_1}, \ldots, \vec{u_k}) \supseteq span(\vec{v_1}, \ldots, \vec{v_m})$$

Use the result above to prove that:

$$\mathbb{R}^3 = span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

$$\mathbb{R}^3 = span(\vec{e_1}, \vec{e_2}, \vec{e_3})$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{e_1}$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \vec{e_1} + \vec{e_2}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \vec{e_1} + \vec{e_2} + \vec{e_3}$$

$$span(\vec{e_1}, \vec{e_2}, \vec{e_3}) \subseteq span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

$$\mathbb{R}^3 = span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

Exercise 23

Determine if the sets of vectors are linearly independent.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ 1 & 3 & 2 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 3 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & -5 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

Linearly independent.

Exercise 25

Determine if the sets of vectors are linearly independent.

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Linearly dependent.

Exercise 27

Determine if the sets of vectors are linearly independent.

$$\begin{bmatrix}3\\4\\5\end{bmatrix}, \begin{bmatrix}6\\7\\8\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Linearly dependent, $\vec{0}$ is linearly dependent to all other vectors.

Exercise 29

Determine if the sets of vectors are linearly independent.

$$\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & 1 & 0 & 0\\ -1 & 1 & 0 & 1 & 0\\ 1 & 0 & 1 & -1 & 0\\ 0 & 1 & -1 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 1 & -1 & 0\\ 0 & 1 & 0 & -1 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & -1 & 0\end{bmatrix}$$
$$= \begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\end{bmatrix}$$

Linearly independent.

Exercise 31

Determine if the sets of vectors are linearly independent.

$$\begin{bmatrix}3\\-1\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\3\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\3\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\1\\3\end{bmatrix}$$

$$\begin{bmatrix}3 & -1 & 1 & -1 & 0\\ -1 & 3 & 1 & -1 & 0\\ 1 & 1 & 3 & 1 & 0\\ -1 & -1 & 1 & 3 & 0\end{bmatrix} = \begin{bmatrix}2 & 2 & 2 & -2 & 0\\ 0 & 4 & 4 & 0 & 0\\ 0 & 0 & 4 & 4 & 0\\ -1 & -1 & 1 & 3 & 0\end{bmatrix}$$
$$= \begin{bmatrix}0 & 0 & 4 & 4 & 0\\ 0 & 4 & 4 & 0 & 0\\ 0 & 0 & 4 & 4 & 0\\ -1 & -1 & 1 & 3 & 0\end{bmatrix}$$
$$= \begin{bmatrix}-1 & -1 & 1 & 3 & 0\\ 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Linearly dependent, infinite solutions.

Exercise 42a

If the columns of an $n \times n$ matrix A are linearly independent as vectors in $\mathbb{R}^n$, what is the rank of A? Explain. The rank of A is n because there exists a non-trivial solution. It is possible to reduce A to reduced row echelon form with no row of 0s at the bottom.

Exercise 44

Prove two vectors are linearly dependent if and only if one is a scalar multiple of the other.
Suppose $\vec{u}$ is a scalar multiple of $\vec{v}$.

$$\begin{aligned} \vec{u} &= c\vec{v} \\ c\vec{v} - \vec{u} &= \vec{0} \end{aligned}$$

Thus, $\vec{u}$ and $\vec{v}$ are linearly dependent.

Exercise 46

Prove that every subset of a linearly independent set is linearly independent. Let $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ be a linearly independent set.

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} \neq \vec{0}$$

By Theorem 2.5, $\vec{v}, \vec{v_2}, \ldots, \vec{v_n}$ are linearly dependent if and only if at least one of the vectors can be expressed as a linear combination of the others. If no such set of scalars exist, then no subset of $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ can be linearly dependent. Thus, every subset of a linearly independent set is linearly independent.

Exercise 47

Suppse that $S = \{\vec{v_1}, \ldots, \vec{v_k}, \vec{v}\}$ is a set of vectors in some $\mathbb{R}^n$ and that $\vec{v}$ is a linear combination of $\vec{v_1}, \ldots, \vec{v_k}$. If $S' = \{\vec{v_1}, \ldots, \vec{v_k}\}$, prove that $span(S) = span(S')$.

$$\vec{v} = c_1\vec{v_1} + \cdots + c_k\vec{v_k}$$

If a vector $\vec{w} \in span(S)$:

$$\begin{aligned} \vec{w} &= d_1\vec{v_1} + \cdots + d_k\vec{v_k} + d\vec{v} \\ &= d_1\vec{v_1} + \cdots + d_k\vec{v_k} + d(c_1\vec{v_1} + \cdots + c_k\vec{v_k}) \\ &= d_1\vec{v_1} + \cdots + d_k\vec{v_k} + dc_1\vec{v_1} + \cdots + dc_k\vec{v_k}) \\ &= (d_1 + dc_1)\vec{v_1} + \cdots + (d_k + dc_k)\vec{v_k} \\ \therefore \vec{w} &\in span(S') \\ span(S) &\subseteq span(S') \end{aligned}$$

If a vector $\vec{x} \in span(S')$:

$$\begin{aligned} \vec{x} &= a_1\vec{v_1} + \cdots + a_k\vec{v_k} + 0\vec{v} \\ \therefore \vec{x} &\in span(S) \\ span(S') &\subseteq span(S) \\ \therefore span(S) &= span(S') \end{aligned}$$

Exercise 48

Let $\{\vec{v_1}, \ldots, \vec{v_k}\}$ be a linearly independent set of vectors in $\mathbb{R}^n$, and let $\vec{v}$ be a vector in $\mathbb{R}^n$. Suppose that $\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k}$ with $c_1 \neq 0$. Prove that $\{\vec{v}, \vec{v_2}, \ldots, \vec{v_k}\}$ is linearly independent.

$$\begin{aligned}
\vec{v} &= c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} \\
\vec{0} &= c\vec{v} + c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} \\
c_1\vec{v_1} &= c\vec{v} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} \\
c_1 &\neq 0 \\
\vec{0} &\neq c\vec{v} + c_2\vec{v_2} + \cdots + c_k\vec{v_k}
\end{aligned}$$

Therefore, $\{\vec{v}, \vec{v_2}, \vec{v_k}\}$ is linearly independent.

Section 3.1

Let:

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}, B = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}, E = \begin{bmatrix} 4 & 2 \end{bmatrix}, F = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Exercise 1

Compute the indicated matrices.

$$\begin{aligned}
A + 2D &= \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} + 2\begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 3 & -6 \\ -5 & 7 \end{bmatrix}
\end{aligned}$$

Exercise 3

Compute the indicated matrices.

$$B - C$$

Undefined.

Exercise 5

Compute the indicated matrices.

$$AB = \begin{bmatrix} 12 & -6 & 3 \\ -4 & 12 & 14 \end{bmatrix}$$

Exercise 7

Compute the indicated matrices.

$$\begin{aligned} D + BC &= \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 21 & 26 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 \\ 19 & 27 \end{bmatrix} \end{aligned}$$

Exercise 9

Compute the indicated matrices.

$$\begin{aligned} E(AF) &= \begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ 11 \end{bmatrix} \\ &= \begin{bmatrix} 10 \end{bmatrix} \end{aligned}$$

Exercise 11

Compute the indicated matrices.

$$FE = \begin{bmatrix} -4 & -2 \\ 8 & 4 \end{bmatrix}$$

Exercise 13

Compute the indicated matrices.

$$\begin{aligned} B^T C^T - (CB)^T &= \begin{bmatrix} 4 & 0 \\ -2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 2 & 7 \\ 12 & 2 & 15 \\ 20 & 2 & 23 \end{bmatrix}^T \\ &= \begin{bmatrix} 4 & 12 & 20 \\ 2 & 2 & 2 \\ 7 & 15 & 23 \end{bmatrix} - \begin{bmatrix} 4 & 12 & 20 \\ 2 & 2 & 2 \\ 7 & 15 & 23 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Exercise 15

Compute the indicated matrices.

$$\begin{aligned} A^3 &= \begin{bmatrix} 9 & 0 \\ -8 & 25 \end{bmatrix} A \\ &= \begin{bmatrix} 27 & 0 \\ -49 & 125 \end{bmatrix} \end{aligned}$$

Exercise 17

Given an example of a nonzero 2×2 matrix A such that $A^2 = 0$.

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Exercise 21

Write the given system of linear equations as a matrix equation of the form $A\vec{x} = \vec{b}$.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 - 5x_3 &= 4 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

Exercise 22

Write the given system of linear equations as a matrix equation of the form $A\vec{x} = \vec{b}$.

$$\begin{aligned} -x_1 + 2x_3 &= 1 \\ x_1 - x_2 &= -2 \\ x_2 + x_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

Exercise 35

Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$. Compute $A^2, A^3, \ldots, A^7$. What is A^{2015}?

$$\begin{aligned} A^2 &= \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \\ A^3 &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ A^4 &= \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \\ A^5 &= \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \\ A^6 &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = I_2 \\ A^7 &= A^1 \\ A^{2015} &= A^{6(370)+5} \\ &= (A^6)^{370} A^5 \\ &= (I_2)^{370} A^5 \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

Exercise 36

Let:

$$B = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Find B^{2015}.

$$
\begin{aligned}
B^2 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
B^3 &= \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \\
B^4 &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\
B^8 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \\
B^{2015} &= B^{8(251)+7} \\
&= (I_2)^{250} B^7 \\
&= B^3 B^4 \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}
\end{aligned}
$$

Exercise 37

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find a formula for $A^n (n \geq 1)$ and verify your formula using mathematical induction.

$$
\begin{aligned}
A^2 &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\
A^3 &= \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \\
A^n &= \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Base Case: $A^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Induction Step: Assume it is true for n. Prove it is true for $n+1$:

$$
\begin{aligned}
A^{n+1} &= A^n A \\
&= \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Exercise 38

Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Show that $A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$.

$$\begin{aligned}
A \times A &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\theta - \sin^2\theta & -\cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta + \cos\theta\sin\theta & -\sin^2\theta + \cos^2\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta+\theta) & -\sin(\theta+\theta) \\ \sin(\theta+\theta) & \cos(\theta+\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}
\end{aligned}$$

Prove by mathematical induction that:

$$A^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix} \qquad \text{for } n \geq 1$$

Base Case ($n = 1$): Obvious.
Induction Step: Assume it is true for n. Prove it is true for $n+1$:

$$\begin{aligned}
A^{n+1} &= A^n A \\
&= \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta) & \cos(n\theta)(-\sin(\theta)) - \sin(n\theta)\cos(\theta) \\ \sin(n\theta)\cos(\theta) + \cos(n\theta)\sin(\theta) & -\sin(n\theta)\sin(\theta) + \cos(n\theta)\cos(\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos(n\theta+\theta) & -\sin(n\theta+\theta) \\ \sin(n\theta+\theta) & \cos(n\theta+\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos((n+1)\theta) & -\sin((n+1)\theta) \\ \sin((n+1)\theta) & \cos((n+1)\theta) \end{bmatrix}
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech