

# Linear Algebra: Homework 5

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August 2016 - December 2016

## Section 2.3

### Exercise 1

Determine if the vector  $\vec{v}$  is a linear combination of the remaining vectors.

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 3 \\ -1 & -12 & \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 3 \\ -1 & 05 & \end{bmatrix} \end{aligned}$$

Yes

### Exercise 3

Determine if the vector  $\vec{v}$  is a linear combination of the remaining vectors.

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix} \end{aligned}$$

No

### Exercise 5

Determine if the vector  $\vec{v}$  is a linear combination of the remaining vectors.

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \vec{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 4 \\ 0 & 1 & 1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}
\end{aligned}$$

Yes

### Exercise 7

Determine if the vector  $\vec{v}$  is in the span of the columns of the matrix  $A$ .

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{aligned}
A\vec{b} &= \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 5 \\ 1 & 0 & -4 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2 & 9 \\ 1 & 0 & -4 \end{bmatrix}
\end{aligned}$$

There is a unique solution for:

$$1a + 2b = 5$$

$$3a + 4b = 6$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{9}{2} \end{bmatrix}$$

**Exercise 9**

Show that  $\mathbb{R}^2 = \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$

$$\begin{aligned} \begin{bmatrix} 1 & 1 & a \\ 1 & -1 & b \end{bmatrix} &= \begin{bmatrix} 1 & 1 & a \\ 2 & 0 & a+b \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & a \\ 1 & 0 & \frac{a+b}{2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & a - \frac{a+b}{2} \\ 1 & 0 & \frac{a+b}{2} \end{bmatrix} \\ x &= a - \frac{a+b}{2} \\ y &= \frac{a+b}{2} \end{aligned}$$

Any point in  $\mathbb{R}^2$  can be described by a linear combination of the two vectors using the constants described above.

**Exercise 11**

Show that  $\mathbb{R}^3 = \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 1 & 0 & 1 & c \end{bmatrix} &= \begin{bmatrix} 1 & 0 & -1 & a-b \\ 0 & 1 & 1 & b \\ 1 & 0 & 1 & c \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 & a-b+c \\ 0 & 1 & 1 & b \\ 1 & 0 & 1 & c \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & \frac{a-b+c}{2} \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & c - \frac{a-b+c}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & \frac{a-b+c}{2} \\ 0 & 1 & 0 & b - (c - \frac{a-b+c}{2}) \\ 0 & 0 & 1 & c - \frac{a-b+c}{2} \end{bmatrix} \\ x &= \frac{a-b+c}{2} \\ y &= b - (c - \frac{a-b+c}{2}) \\ z &= c - \frac{a-b+c}{2} \end{aligned}$$

Any point in  $\mathbb{R}^3$  can be described by a linear combination of the two vectors using the constants described above.

**Exercise 13b**

Describe the span of the given vectors algebraically.

$$\begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{aligned}
s \begin{bmatrix} 2 \\ -4 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \end{bmatrix} &= \begin{bmatrix} x \\ y \end{bmatrix} \\
2s - t &= x \\
-4s + 2t &= y \\
\begin{bmatrix} 2 & -1 & x \\ -4 & 2 & y \end{bmatrix} &= \begin{bmatrix} 2 & -1 & x \\ 0 & 0 & y + 2x \end{bmatrix} \\
y + 2x &= 0 \\
y &= -2x
\end{aligned}$$

The span of the two vectors describes the set of all vectors parallel and antiparallel to the given vectors, which line on the line  $y = -2x$ .

### Exercise 15b

Describe the span of the given vectors algebraically.

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{aligned}
s \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} &= \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
s + 3t &= x \\
2s + 2t &= y \\
-t &= z \\
\begin{bmatrix} 1 & 3 & x \\ 2 & 2 & y \\ 0 & -1 & z \end{bmatrix} &= \begin{bmatrix} 1 & 0 & x + 3z \\ 2 & 0 & y + 2z \\ 0 & -1 & z \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & x + 3z \\ 0 & 0 & y + 2z - 2(x + 3z) \\ 0 & -1 & z \end{bmatrix} \\
0 &= y + 2z - 2x - 6z \\
&= -2x + y - 4z
\end{aligned}$$

The span of the vectors describes a plane with the equation  $0 = -2x + y - 4z$ .

### Exercise 17

The general equation of the plane that contains the points  $(1,0,3), (-1,1,-3)$ , and the origin is of the form  $ax + by + cz = 0$ . Solve for  $a, b, c$ .

$$\begin{aligned} \begin{bmatrix} 1 & -1 & 0 & x \\ 0 & 1 & 0 & y \\ 3 & -3 & 0 & z \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & x+y \\ 0 & 1 & 0 & y \\ 1 & -1 & 0 & \frac{z}{3} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & x+y \\ 0 & 1 & 0 & y \\ 1 & 0 & 0 & \frac{z}{3} + y \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & x+y \\ 0 & 1 & 0 & y \\ 0 & 0 & 0 & \frac{z}{3} + y - (x+y) \end{bmatrix} \\ 0 &= \frac{z}{3} + y - x - y \\ &= -x + \frac{z}{3} \\ &= -3x + z \\ a &= -3 \\ b &= 0 \\ c &= 1 \end{aligned}$$

### Exercise 18

Prove that  $\vec{u}, \vec{v}, \vec{w}$  are all in  $\text{span}(\vec{u}, \vec{v}, \vec{w})$ .

$$\begin{aligned} \vec{u} &= 1\vec{u} + 0\vec{v} + 0\vec{w} \\ \vec{v} &= 0\vec{u} + 1\vec{v} + 0\vec{w} \\ \vec{w} &= 0\vec{u} + 0\vec{v} + 1\vec{w} \\ \vec{u}, \vec{v}, \vec{w} &\in \text{span}(\vec{u}, \vec{v}, \vec{w}) \end{aligned}$$

### Exercise 19

$$\begin{aligned} \vec{u} &= 1\vec{u} + 0(\vec{u} + \vec{v}) + 0(\vec{u} + \vec{v} + \vec{w}) \\ \vec{v} &= -1\vec{u} + 1(\vec{u} + \vec{v}) + 0(\vec{u} + \vec{v} + \vec{w}) \\ \vec{w} &= 0\vec{u} - 1(\vec{u} + \vec{v}) + 1(\vec{u} + \vec{v} + \vec{w}) \\ \vec{u}, \vec{v}, \vec{w} &\in \text{span}(\vec{u}, \vec{u} + \vec{v}, \vec{u} + \vec{v} + \vec{w}) \end{aligned}$$

### Exercise 20

Prove that if  $\vec{u}_1, \dots, \vec{u}_m \in \mathbb{R}^n$  where  $S = \{\vec{u}_1, \dots, \vec{u}_k\}$  and  $T = \{\vec{u}_1, \dots, \vec{u}_k, \vec{u}_{k+1}, \dots, \vec{u}_m\}$  that  $\text{span}(S) \subseteq \text{span}(T)$ .

$$\begin{aligned} c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_k\vec{u}_k &\in \text{span}(S) \\ c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_k\vec{u}_k &= c_1\vec{u}_1 + \dots + c_k\vec{u}_k + 0\vec{u}_{k+1} + \dots + 0\vec{u}_m \\ c_1\vec{u}_1 + \dots + c_k\vec{u}_k + 0\vec{u}_{k+1} + \dots + 0\vec{u}_m &\in \text{span}(T) \\ \text{span}(S) &\subseteq \text{span}(T) \end{aligned}$$

Deduce if  $\mathbb{R}^n = \text{span}(S)$ , then  $\mathbb{R}^n = \text{span}(T)$ .

$$\begin{aligned}\mathbb{R}^n &= \text{span}(S) \\ \text{span}(S) &\subseteq \text{span}(T) \\ \text{span}(T) &\supseteq \mathbb{R}^n \\ \text{span}(T) &= \mathbb{R}^n\end{aligned}$$

### Exercise 21

Suppose  $\vec{w}$  is a linear combination of  $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$  and each  $\vec{u}_i$  is a linear combination of  $\vec{v}_1, \dots, \vec{v}_m$ . Prove  $\vec{w}$  is a linear combination of  $\vec{v}_1, \dots, \vec{v}_m$ .

$$\begin{aligned}\vec{w} &= \sum_{i=1}^k c_i u_i \\ &= \sum_{i=1}^k c_i \left( \sum_{j=1}^m d_{ij} \vec{v}_j \right) \\ &= \sum_{i=1}^k \left( \sum_{j=1}^m c_i d_{ij} \vec{v}_j \right) \\ &= \sum_{j=1}^m \left( \sum_{i=1}^k c_i d_{ij} \right) \vec{v}_j \\ w &\in \text{span}(\vec{v}_1, \dots, \vec{v}_m)\end{aligned}$$

Also suppose each  $\vec{v}_j$  is a linear combination of  $\vec{u}_1, \dots, \vec{u}_k$ . Prove  $\text{span}(\vec{u}_1, \dots, \vec{u}_k) = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$ . Above we proved:

$$\text{span}(\vec{u}_1, \dots, \vec{u}_k) \subseteq \text{span}(\vec{v}_1, \dots, \vec{v}_m)$$

Therefore:

$$\text{span}(\vec{u}_1, \dots, \vec{u}_k) \supseteq \text{span}(\vec{v}_1, \dots, \vec{v}_m)$$

Use the result above to prove that:

$$\mathbb{R}^3 = \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

$$\mathbb{R}^3 = \text{span}(\vec{e}_1, \vec{e}_2, \vec{e}_3)$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{e}_1$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \vec{e}_1 + \vec{e}_2$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \vec{e}_1 + \vec{e}_2 + \vec{e}_3$$

$$\text{span}(\vec{e}_1, \vec{e}_2, \vec{e}_3) \subseteq \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

$$\mathbb{R}^3 = \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

### Exercise 23

Determine if the sets of vectors are linearly independent.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ 1 & 3 & 2 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 3 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & -5 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

Linearly independent.

### Exercise 25

Determine if the sets of vectors are linearly independent.

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Linearly dependent.

**Exercise 27**

Determine if the sets of vectors are linearly independent.

$$\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Linearly dependent,  $\vec{0}$  is linearly dependent to all other vectors.

**Exercise 29**

Determine if the sets of vectors are linearly independent.

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Linearly independent.

**Exercise 31**

Determine if the sets of vectors are linearly independent.

$$\begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 1 & -1 & 0 \\ -1 & 3 & 1 & -1 & 0 \\ 1 & 1 & 3 & 1 & 0 \\ -1 & -1 & 1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 & -2 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ -1 & -1 & 1 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 4 & 4 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ -1 & -1 & 1 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Linearly dependent, infinite solutions.



### Exercise 42a

If the columns of an  $n \times n$  matrix  $A$  are linearly independent as vectors in  $\mathbb{R}^n$ , what is the rank of  $A$ ? Explain. The rank of  $A$  is  $n$  because there exists a non-trivial solution. It is possible to reduce  $A$  to reduced row echelon form with no row of 0s at the bottom.

### Exercise 44

Prove two vectors are linearly dependent if and only if one is a scalar multiple of the other. Suppose  $\vec{u}$  is a scalar multiple of  $\vec{v}$ .

$$\begin{aligned}\vec{u} &= c\vec{v} \\ c\vec{v} - \vec{u} &= \vec{0}\end{aligned}$$

Thus,  $\vec{u}$  and  $\vec{v}$  are linearly dependent.

### Exercise 46

Prove that every subset of a linearly independent set is linearly independent. Let  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  be a linearly independent set.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n \neq \vec{0}$$

By Theorem 2.5,  $\vec{v}, \vec{v}_2, \dots, \vec{v}_n$  are linearly dependent if and only if at least one of the vectors can be expressed as a linear combination of the others. If no such set of scalars exist, then no subset of  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  can be linearly dependent. Thus, every subset of a linearly independent set is linearly independent.

### Exercise 47

Suppose that  $S = \{\vec{v}_1, \dots, \vec{v}_k, \vec{v}\}$  is a set of vectors in some  $\mathbb{R}^n$  and that  $\vec{v}$  is a linear combination of  $\vec{v}_1, \dots, \vec{v}_k$ . If  $S' = \{\vec{v}_1, \dots, \vec{v}_k\}$ , prove that  $\text{span}(S) = \text{span}(S')$ .

$$\vec{v} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$$

If a vector  $\vec{w} \in \text{span}(S)$ :

$$\begin{aligned}\vec{w} &= d_1\vec{v}_1 + \dots + d_k\vec{v}_k + d\vec{v} \\ &= d_1\vec{v}_1 + \dots + d_k\vec{v}_k + d(c_1\vec{v}_1 + \dots + c_k\vec{v}_k) \\ &= d_1\vec{v}_1 + \dots + d_k\vec{v}_k + dc_1\vec{v}_1 + \dots + dc_k\vec{v}_k \\ &= (d_1 + dc_1)\vec{v}_1 + \dots + (d_k + dc_k)\vec{v}_k \\ &\therefore \vec{w} \in \text{span}(S') \\ \text{span}(S) &\subseteq \text{span}(S')\end{aligned}$$

If a vector  $\vec{x} \in \text{span}(S')$ :

$$\begin{aligned}\vec{x} &= a_1\vec{v}_1 + \dots + a_k\vec{v}_k + 0\vec{v} \\ &\therefore \vec{x} \in \text{span}(S) \\ \text{span}(S') &\subseteq \text{span}(S) \\ &\therefore \text{span}(S) = \text{span}(S')\end{aligned}$$

### Exercise 48

Let  $\{\vec{v}_1, \dots, \vec{v}_k\}$  be a linearly independent set of vectors in  $\mathbb{R}^n$ , and let  $\vec{v}$  be a vector in  $\mathbb{R}^n$ . Suppose that  $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$  with  $c_1 \neq 0$ . Prove that  $\{\vec{v}, \vec{v}_2, \dots, \vec{v}_k\}$  is linearly independent.

$$\begin{aligned}\vec{v} &= c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k \\ \vec{0} &= c\vec{v} + c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k \\ c_1\vec{v}_1 &= c\vec{v} + c_2\vec{v}_2 + \dots + c_k\vec{v}_k \\ c_1 &\neq 0 \\ \vec{0} &\neq c\vec{v} + c_2\vec{v}_2 + \dots + c_k\vec{v}_k\end{aligned}$$

Therefore,  $\{\vec{v}, \vec{v}_2, \vec{v}_k\}$  is linearly independent.

## Section 3.1

Let:

$$\begin{aligned}A &= \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}, B = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \\ D &= \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}, E = [4 \ 2], F = \begin{bmatrix} -1 \\ 2 \end{bmatrix}\end{aligned}$$

### Exercise 1

Compute the indicated matrices.

$$\begin{aligned}A + 2D &= \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} + 2 \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -6 \\ -5 & 7 \end{bmatrix}\end{aligned}$$

### Exercise 3

Compute the indicated matrices.

$$B - C$$

Undefined.

### Exercise 5

Compute the indicated matrices.

$$AB = \begin{bmatrix} 12 & -6 & 3 \\ -4 & 12 & 14 \end{bmatrix}$$

**Exercise 7**

Compute the indicated matrices.

$$\begin{aligned} D + BC &= \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 21 & 26 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 \\ 19 & 27 \end{bmatrix} \end{aligned}$$

**Exercise 9**

Compute the indicated matrices.

$$\begin{aligned} E(AF) &= [4 \ 2] \begin{bmatrix} -3 \\ 11 \end{bmatrix} \\ &= [10] \end{aligned}$$

**Exercise 11**

Compute the indicated matrices.

$$FE = \begin{bmatrix} -4 & -2 \\ 8 & 4 \end{bmatrix}$$

**Exercise 13**

Compute the indicated matrices.

$$\begin{aligned} B^T C^T - (CB)^T &= \begin{bmatrix} 4 & 0 \\ -2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 2 & 7 \\ 12 & 2 & 15 \\ 20 & 2 & 23 \end{bmatrix}^T \\ &= \begin{bmatrix} 4 & 12 & 20 \\ 2 & 2 & 2 \\ 7 & 15 & 23 \end{bmatrix} - \begin{bmatrix} 4 & 12 & 20 \\ 2 & 2 & 2 \\ 7 & 15 & 23 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

**Exercise 15**

Compute the indicated matrices.

$$\begin{aligned} A^3 &= \begin{bmatrix} 9 & 0 \\ -8 & 25 \end{bmatrix} A \\ &= \begin{bmatrix} 27 & 0 \\ -49 & 125 \end{bmatrix} \end{aligned}$$

**Exercise 17**

Given an example of a nonzero  $2 \times 2$  matrix  $A$  such that  $A^2 = 0$ .

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

**Exercise 21**

Write the given system of linear equations as a matrix equation of the form  $A\vec{x} = \vec{b}$ .

$$\begin{aligned}x_1 - 2x_2 + 3x_3 &= 0 \\2x_1 + x_2 - 5x_3 &= 4\end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

**Exercise 22**

Write the given system of linear equations as a matrix equation of the form  $A\vec{x} = \vec{b}$ .

$$\begin{aligned}-x_1 + 2x_3 &= 1 \\x_1 - x_2 &= -2 \\x_2 + x_3 &= -1\end{aligned}$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

**Exercise 35**

Let  $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ . Compute  $A^2, A^3, \dots, A^7$ . What is  $A^{2015}$ ?

$$\begin{aligned}A^2 &= \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \\A^3 &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\A^4 &= \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \\A^5 &= \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \\A^6 &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = I_2 \\A^7 &= A^1 \\A^{2015} &= A^{6(370)+5} \\&= (A^6)^{370} A^5 \\&= (I_2)^{370} A^5 \\&= \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\end{aligned}$$

**Exercise 36**

Let:

$$B = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Find  $B^{2015}$ .

$$\begin{aligned} B^2 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ B^3 &= \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \\ B^4 &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ B^8 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \\ B^{2015} &= B^{8(251)+7} \\ &= (I_2)^{250} B^7 \\ &= B^3 B^4 \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

### Exercise 37

Let  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ . Find a formula for  $A^n$  ( $n \geq 1$ ) and verify your formula using mathematical induction.

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\ A^3 &= \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \\ A^n &= \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Base Case:  $A^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ . Induction Step: Assume it is true for  $n$ . Prove it is true for  $n + 1$ :

$$\begin{aligned} A^{n+1} &= A^n A \\ &= \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

### Exercise 38

Let  $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ . Show that  $A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$ .

$$\begin{aligned} A \times A &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \theta - \sin^2 \theta & -\cos \theta \sin \theta - \sin \theta \cos \theta \\ \sin \theta \cos \theta + \cos \theta \sin \theta & -\sin^2 \theta + \cos^2 \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta + \theta) & -\sin(\theta + \theta) \\ \sin(\theta + \theta) & \cos(\theta + \theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix} \end{aligned}$$

Prove by mathematical induction that:

$$A^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix} \quad \text{for } n \geq 1$$

Base Case ( $n = 1$ ): Obvious.

Induction Step: Assume it is true for  $n$ . Prove it is true for  $n + 1$ :

$$\begin{aligned} A^{n+1} &= A^n A \\ &= \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(n\theta) \cos(\theta) - \sin(n\theta) \sin(\theta) & \cos(n\theta)(-\sin(\theta)) - \sin(n\theta) \cos(\theta) \\ \sin(n\theta) \cos(\theta) + \cos(n\theta) \sin(\theta) & -\sin(n\theta) \sin(\theta) + \cos(n\theta) \cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos(n\theta + \theta) & -\sin(n\theta + \theta) \\ \sin(n\theta + \theta) & \cos(n\theta + \theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos((n + 1)\theta) & -\sin((n + 1)\theta) \\ \sin((n + 1)\theta) & \cos((n + 1)\theta) \end{bmatrix} \end{aligned}$$

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