Linear Algebra: Homework 4

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Section 2.2

Exercise 1

Determine whether the given matrix is in row echelon form. If it is, state whether it is also in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

No.

Exercise 3

Determine whether the given matrix is in row echelon form. If it is, state whether it is also in reduced row echelon form.

$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Yes, and it is in reduced row echelon form.

Exercise 5

Determine whether the given matrix is in row echelon form. If it is, state whether it is also in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & 1 \end{bmatrix}$$

No.

Exercise 7

Determine whether the given matrix is in row echelon form. If it is, state whether it is also in reduced row echelon form.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

No.

Exercise 9

Determine whether the given matrix is in row echelon form. If it is, state whether it is also in reduced row echelon form.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

No.

Exercise 11

Use elementary row operations to reduce the given matrix to row echelon form and reduced row echelon form.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$REF = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 - R_2 \rightarrow R_1$$

$$RREF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 13

Use elementary row operations to reduce the given matrix to row echelon form and reduced row echelon form.

$$A = \begin{bmatrix} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -1 \end{bmatrix}$$

$$-2R_2 + R_3 \rightarrow R_3$$

$$= \begin{bmatrix} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$2R_1 \rightarrow R_1$$

$$3R_2 \rightarrow R_2$$

$$= \begin{bmatrix} 6 & -4 & -2 \\ 6 & -3 & -3 \\ 0 & -1 & 1 \end{bmatrix}$$

$$R_2 - R_1 \rightarrow R_2$$

$$= \begin{bmatrix} 6 & -4 & -2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$R_3 - R_2 \rightarrow R_3$$

$$REF = \begin{bmatrix} 6 & -4 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 - R_2 \rightarrow R_1$$

$$= \begin{bmatrix} 6 & 0 & -6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{6}R_1 \rightarrow R_1$$

$$RREF = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Exercise 17

Show that the given matrices are row equivalent and find a sequence of elementary row operations that will convert A into B.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ -2R_1 + R_2 &\to R_2 \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \\ -\frac{1}{2}R_1 &\to R_1 \\ &= \begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & 0 \end{bmatrix} \\ -6R_2 + R_1 &\to R_1 \\ &= \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} = B \end{aligned}$$

Exercise 23

What is the rank of each of the matrices in Exercises 1-8?

1. Rank: 3
2. Rank: 2
3. Rank: 2
4. Rank: 0
5. Rank: 2
6. Rank: 3
7. Rank: 3
8. Rank: 3

Exercise 25

Solve the given system of equations:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 9 \\ 2x_1 - x_2 + x_3 &= 0 \\ 4x_1 - x_2 + x_3 &= 4 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{bmatrix}$$

$$-2R_2 + R_3 \to R_3$$

$$= \begin{bmatrix} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 0 & 1 & -1 & 4 \end{bmatrix}$$

$$-2R_1 + R_2 \to R_2$$

$$= \begin{bmatrix} 1 & 2 & -3 & 9 \\ 0 & -5 & 7 & -18 \\ 0 & 1 & -1 & 4 \end{bmatrix}$$

$$5R_3 + R_2 \to R_2$$

$$R_2 \leftrightarrow R_3$$

$$= \begin{bmatrix} 1 & 2 & -3 & 9 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\frac{1}{2}R_3 \to R_3$$

$$= \begin{bmatrix} 1 & 2 & -3 & 9 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$-2R_2 + R_1 \to R_1$$

$$R_3 + R_2 \to R_2$$

$$= \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$R_1 + R_3 \to R_1$$

$$= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$x_1 = 2$$

$$x_2 = 3$$

$$x_3 = 1$$

Exercise 27

Solve the given system of equations:

$$\begin{aligned} x_1 - 3x_2 - 2x_3 &= 0 \\ -x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + 4x_2 + 6x_3 &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & -3 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 4 & 6 & 0 \end{bmatrix}$$

$$2R_2 + R_3 \to R_3$$

$$= \begin{bmatrix} 1 & -3 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 8 & 8 & 0 \end{bmatrix}$$

$$\frac{1}{8}R_3 \to R_3$$

$$= \begin{bmatrix} 1 & -3 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$-2R_3 + R_2 \to R_2$$

$$= \begin{bmatrix} 1 & -3 & -2 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$R_2 + R_1 \to R_1$$

$$= \begin{bmatrix} 0 & -3 & -3 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = y = z$$

Exercise 29

Solve the given system of equations:

$$\begin{aligned} 2r + s &= 3 \\ 4r + s &= 7 \\ 2r + 5s &= -1 \end{aligned}$$

$$\begin{aligned} A &= \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ 2 & 5 & -1 \end{bmatrix} \\ R_3 - R_1 &\to R_3 \\ &= \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ 0 & 4 & -4 \end{bmatrix} \\ \frac{1}{4}R_3 &\to R_3 \\ -2R_1 + R_2 &\to R_2 \\ &= \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} R_1 - R_3 &\to R_1 \\ R_3 + R_2 &\to R_2 \\ R_3 &\leftrightarrow R_2 \\ &= \begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ \frac{1}{2}R_1 &\to R_1 \\ &= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ r &= 2 \\ s &= -1 \end{aligned}$$

Exercise 31

Solve the given system of equations:

$$\begin{aligned} \frac{1}{2}x_1 + x_2 \quad -x_3 - 6x_4 \qquad &= 2 \\ \frac{1}{6}x_1 + \frac{1}{2}x_2 \qquad - 3x_4 \ +x_5 &= -1 \\ \frac{1}{3}x_1 \qquad -2x_3 \qquad -4x_5 &= 8 \end{aligned}$$

$$A = \begin{bmatrix} \frac{1}{2} & 1 & -1 & -6 & 0 & 2 \\ \frac{1}{6} & \frac{1}{2} & 0 & -3 & 1 & -1 \\ \frac{1}{3} & 0 & -2 & 0 & -4 & 8 \end{bmatrix}$$

$$2R_1 \to R_1$$
$$6R_2 \to R_2$$
$$3R_3 \to R_3$$

$$= \begin{bmatrix} 1 & 2 & -2 & -12 & 0 & 4 \\ 1 & 3 & 0 & -18 & 6 & -6 \\ 3 & 0 & -6 & 0 & -12 & 24 \end{bmatrix}$$

$$R_1 - R_2 \to R_2$$
$$3R_1 - R_3 \to R_3$$

$$= \begin{bmatrix} 1 & 2 & -2 & -12 & 0 & 4 \\ 0 & -1 & -2 & 6 & -6 & 10 \\ 0 & 6 & 0 & -36 & 12 & -12 \end{bmatrix}$$

$$6R_2 + R_3 \to R_3$$

$$= \begin{bmatrix} 1 & 2 & -2 & -12 & 0 & 4 \\ 0 & -1 & -2 & 6 & -6 & 10 \\ 0 & 0 & -12 & 0 & -24 & 48 \end{bmatrix}$$

$$\frac{-1}{12}R_3 \to R_3$$
$$2R_2 + R_1 \to R_1$$

$$= \begin{bmatrix} 1 & 0 & -6 & 0 & -12 & 24 \\ 0 & -1 & -2 & 6 & -6 & 10 \\ 0 & 0 & 1 & 0 & 2 & -4 \end{bmatrix}$$

$$6R_3 + R_1 \to R_1$$
$$2R_3 + R_2 \to R_2$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 6 & -2 & 2 \\ 0 & 0 & 1 & 0 & 2 & -4 \end{bmatrix}$$

$$x_1 = 0$$
$$-1x_2 + 6x_4 - 2x_5 = 2$$
$$x_3 + 2x_5 = -4$$

Exercise 33

Solve the given system of equations:

$$\begin{aligned} w+x+2y+z &= 1 \\ w-x-y\ +z &= 0 \\ x+y\quad &= -1 \\ w+x\qquad +z &= 2 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 2 \end{bmatrix}$$

$$R_4 - R_1 \to R_4$$

$$R_2 - R_1 \to R_1$$

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & -2 & -3 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 \end{bmatrix}$$

$$R_1 - R_3 \to R_1$$

$$R_2 + 2R_3 \to R_2$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 \end{bmatrix}$$

$$-2R_2 + R_4 \to R_4$$

$$R_1 + R_2 \to R_2$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$0 = -1$$

System is inconsistent.

Exercise 39

Show that if $ad - bc \neq 0$, then the system

$$ax + by = r$$
$$cx + dy = s$$

has a unique solution.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\det(A)} A \begin{bmatrix} r \\ s \end{bmatrix}$$

$$= \frac{1}{ad - bc} A \begin{bmatrix} r \\ s \end{bmatrix}$$

If $ad - bc \neq 0$, then the system has a unique solution.

Exercise 40

For what values of k, if any, will the systems have no solution, a unique solution, and infinitely many solutions?

$$\begin{aligned} kx + 2y &= 3 \\ 2x - 4y &= -6 \\ A &= \begin{bmatrix} k & 2 & 3 \\ 2 & -4 & -6 \end{bmatrix} \\ &= \begin{bmatrix} k & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} k & 2 & 3 \\ k+1 & 0 & 0 \end{bmatrix} \end{aligned}$$

$k = -1$ will have infinitely many solutions. The system will have a unique solution for all other values.

Exercise 41

For what values of k, if any, will the systems have no solution, a unique solution, and infinitely many solutions?

$$\begin{aligned} x + ky &= 1 \\ kx + y &= 1 \\ A &= \begin{bmatrix} 1 & k & 1 \\ k & 1 & 1 \end{bmatrix} \end{aligned}$$

$k = 1$ will have infinitely many solutions. The system will have a unique solution for all other values.

Exercise 42

For what values of k, if any, will the systems have no solution, a unique solution, and infinitely many solutions?

$$\begin{aligned} x - 2y + 3z &= 2 \\ x + y + z &= k \\ 2x - y + 4z &= k^2 \\ A &= \begin{bmatrix} 1 & -2 & 3 & 2 \\ 1 & 1 & 1 & k \\ 2 & -1 & 4 & k^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & -3 & 2 & 2-k \\ 0 & -3 & 2 & 4-k^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & -3 & 2 & 2-k \\ 0 & 0 & 0 & 2-k+(4-k^2) \end{bmatrix} \\ 2 - k + 4 - k^2 &= 0 \\ k^2 + k - 6 &= 0 \\ (k+3)(k-2) &= 0 \\ k = -3 \quad k &= 2 \end{aligned}$$

The system has infinitely many solutions when $k = 2$, no solutions otherwise.

Exercise 43

For what values of k, if any, will the systems have no solution, a unique solution, and infinitely many solutions?

$$\begin{aligned} x+y+kz&=1\\ x+ky+z&=1\\ kz+y+z&=-2 \end{aligned}$$

$$A=\begin{bmatrix}1&1&k&1\\1&k&1&1\\k&1&1&-2\end{bmatrix}$$

The system has no solution when $k=1$, and a unique solution otherwise.

Exercise 44

Give examples of homogeneous systems of m linear equations in n variables with $m=n$ and with $m>n$ that have infinitely many solutions and a unique solution.
$m=n$: infinite solutions

$$\begin{aligned} x+y&=0\\ 2x+2y&=0 \end{aligned}$$

$m=n$: unique solution

$$\begin{aligned} x+2y&=0\\ 4x+y&=0 \end{aligned}$$

$m>n$: infinite solutions

$$\begin{aligned} x+y&=0\\ 2x+2y&=0\\ 10x+10y&=0 \end{aligned}$$

$m>n$: unique solution

$$\begin{aligned} x+y&=0\\ 3x+1y&=0\\ 10x+3y&=0 \end{aligned}$$

Exercise 45

Find the line of intersection of the given planes.

$$\begin{aligned} 3x+2y+z&=-1\\ 2x-y+4z&=5 \end{aligned}$$

Let $z = t$:

$$\begin{aligned}
3x + 2y &= -1 - t \\
2x - y &= 5 - 4t \\
A &= \begin{bmatrix} 3 & 2 & -1 - t \\ 2 & -1 & 5 - 4t \end{bmatrix} \\
&= \begin{bmatrix} 6 & 4 & -2 - 2t \\ 3 & -1 & 5 - 4t \end{bmatrix} \\
&= \begin{bmatrix} 0 & 6 & 6t - 12 \\ 3 & -1 & 5 - 4t \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & t - 2 \\ 3 & 0 & 3 - 3t \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & t - 2 \\ 1 & 0 & 1 - t \end{bmatrix} \\
l(t) &= \begin{cases} x & = t - 2 \\ y & = 1 - t \\ z & = t \end{cases}
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech