

# Linear Algebra: Homework 4

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## Section 2.2

### Exercise 1

Determine whether the given matrix is in row echelon form. If it is, state whether it is also in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

No.

### Exercise 3

Determine whether the given matrix is in row echelon form. If it is, state whether it is also in reduced row echelon form.

$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Yes, and it is in reduced row echelon form.

### Exercise 5

Determine whether the given matrix is in row echelon form. If it is, state whether it is also in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & 1 \end{bmatrix}$$

No.

### Exercise 7

Determine whether the given matrix is in row echelon form. If it is, state whether it is also in reduced row echelon form.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

No.

**Exercise 9**

Determine whether the given matrix is in row echelon form. If it is, state whether it is also in reduced row echelon form.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

No.

**Exercise 11**

Use elementary row operations to reduce the given matrix to row echelon form and reduced row echelon form.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$REF = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 - R_2 \rightarrow R_1$$

$$RREF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Exercise 13**

Use elementary row operations to reduce the given matrix to row echelon form and reduced row echelon form.

$$A = \begin{bmatrix} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -1 \end{bmatrix}$$

$$-2R_2 + R_3 \rightarrow R_3$$

$$= \begin{bmatrix} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$2R_1 \rightarrow R_1$$

$$3R_2 \rightarrow R_2$$

$$= \begin{bmatrix} 6 & -4 & -2 \\ 6 & -3 & -3 \\ 0 & -1 & 1 \end{bmatrix}$$

$$R_2 - R_1 \rightarrow R_2$$

$$= \begin{bmatrix} 6 & -4 & -2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$R_3 - R_2 \rightarrow R_3$$

$$REF = \begin{bmatrix} 6 & -4 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 - R_2 \rightarrow R_1$$

$$= \begin{bmatrix} 6 & 0 & -6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{6}R_1 \rightarrow R_1$$

$$RREF = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

**Exercise 17**

Show that the given matrices are row equivalent and find a sequence of elementary row operations that will convert  $A$  into  $B$ .

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ -2R_1 + R_2 &\rightarrow R_2 \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \\ -\frac{1}{2}R_1 &\rightarrow R_1 \\ &= \begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & 0 \end{bmatrix} \\ -6R_2 + R_1 &\rightarrow R_1 \\ &= \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} = B \end{aligned}$$

**Exercise 23**

What is the rank of each of the matrices in Exercises 1-8?

1. Rank: 3
2. Rank: 2
3. Rank: 2
4. Rank: 0
5. Rank: 2
6. Rank: 3
7. Rank: 3
8. Rank: 3

**Exercise 25**

Solve the given system of equations:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 9 \\ 2x_1 - x_2 + x_3 &= 0 \\ 4x_1 - x_2 + x_3 &= 4 \end{aligned}$$

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{bmatrix} \\
-2R_2 + R_3 &\rightarrow R_3 \\
&= \begin{bmatrix} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 0 & 1 & -1 & 4 \end{bmatrix} \\
-2R_1 + R_2 &\rightarrow R_2 \\
&= \begin{bmatrix} 1 & 2 & -3 & 9 \\ 0 & -5 & 7 & -18 \\ 0 & 1 & -1 & 4 \end{bmatrix} \\
5R_3 + R_2 &\rightarrow R_2 \\
R_2 &\leftrightarrow R_3 \\
&= \begin{bmatrix} 1 & 2 & -3 & 9 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 2 & 2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}R_3 &\rightarrow R_3 \\
&= \begin{bmatrix} 1 & 2 & -3 & 9 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\
-2R_2 + R_1 &\rightarrow R_1 \\
R_3 + R_2 &\rightarrow R_2 \\
&= \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\
R_1 + R_3 &\rightarrow R_1 \\
&= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\
x_1 &= 2 \\
x_2 &= 3 \\
x_3 &= 1
\end{aligned}$$

### Exercise 27

Solve the given system of equations:

$$\begin{aligned}
x_1 - 3x_2 - 2x_3 &= 0 \\
-x_1 + 2x_2 + x_3 &= 0 \\
2x_1 + 4x_2 + 6x_3 &= 0
\end{aligned}$$

$$\begin{aligned}
A &= \begin{bmatrix} 1 & -3 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 4 & 6 & 0 \end{bmatrix} \\
2R_2 + R_3 &\rightarrow R_3 \\
&= \begin{bmatrix} 1 & -3 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 8 & 8 & 0 \end{bmatrix} \\
\frac{1}{8}R_3 &\rightarrow R_3 \\
&= \begin{bmatrix} 1 & -3 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\
-2R_3 + R_2 &\rightarrow R_2 \\
&= \begin{bmatrix} 1 & -3 & -2 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\
R_2 + R_1 &\rightarrow R_1 \\
&= \begin{bmatrix} 0 & -3 & -3 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
x &= y = z
\end{aligned}$$

**Exercise 29**

Solve the given system of equations:

$$\begin{aligned}2r + s &= 3 \\4r + s &= 7 \\2r + 5s &= -1\end{aligned}$$

$$\begin{aligned}A &= \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ 2 & 5 & -1 \end{bmatrix} \\R_3 - R_1 &\rightarrow R_3 \\&= \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ 0 & 4 & -4 \end{bmatrix} \\\frac{1}{4}R_3 &\rightarrow R_3 \\-2R_1 + R_2 &\rightarrow R_2 \\&= \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}R_1 - R_3 &\rightarrow R_1 \\R_3 + R_2 &\rightarrow R_2 \\R_3 &\leftrightarrow R_1 \\&= \begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\\frac{1}{2}R_1 &\rightarrow R_1 \\&= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\r &= 2 \\s &= -1\end{aligned}$$

**Exercise 31**

Solve the given system of equations:

$$\begin{aligned}\frac{1}{2}x_1 + x_2 - x_3 - 6x_4 &= 2 \\\frac{1}{6}x_1 + \frac{1}{2}x_2 - 3x_4 + x_5 &= -1 \\\frac{1}{3}x_1 - 2x_3 - 4x_5 &= 8\end{aligned}$$

$$A = \begin{bmatrix} \frac{1}{2} & 1 & -1 & -6 & 0 & 2 \\ \frac{1}{6} & \frac{1}{2} & 0 & -3 & 1 & -1 \\ \frac{1}{3} & 0 & -2 & 0 & -4 & 8 \end{bmatrix}$$

$$2R_1 \rightarrow R_1$$

$$6R_2 \rightarrow R_2$$

$$3R_3 \rightarrow R_3$$

$$= \begin{bmatrix} 1 & 2 & -2 & -12 & 0 & 4 \\ 1 & 3 & 0 & -18 & 6 & -6 \\ 3 & 0 & -6 & 0 & -12 & 24 \end{bmatrix}$$

$$R_1 - R_2 \rightarrow R_2$$

$$3R_1 - R_3 \rightarrow R_3$$

$$= \begin{bmatrix} 1 & 2 & -2 & -12 & 0 & 4 \\ 0 & -1 & -2 & 6 & -6 & 10 \\ 0 & 6 & 0 & -36 & 12 & -12 \end{bmatrix}$$

$$6R_2 + R_3 \rightarrow R_3$$

$$= \begin{bmatrix} 1 & 2 & -2 & -12 & 0 & 4 \\ 0 & -1 & -2 & 6 & -6 & 10 \\ 0 & 0 & -12 & 0 & -24 & 48 \end{bmatrix}$$

$$\frac{-1}{12}R_3 \rightarrow R_3$$

$$2R_2 + R_1 \rightarrow R_1$$

$$= \begin{bmatrix} 1 & 0 & -6 & 0 & -12 & 24 \\ 0 & -1 & -2 & 6 & -6 & 10 \\ 0 & 0 & 1 & 0 & 2 & -4 \end{bmatrix}$$

$$6R_3 + R_1 \rightarrow R_1$$

$$2R_3 + R_2 \rightarrow R_2$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 6 & -2 & 2 \\ 0 & 0 & 1 & 0 & 2 & -4 \end{bmatrix}$$

$$x_1 = 0$$

$$-1x_2 + 6x_4 - 2x_5 = 2$$

$$x_3 + 2x_5 = -4$$

### Exercise 33

Solve the given system of equations:

$$w+x+2y+z=1$$

$$w-x-y+z=0$$

$$x+y=-1$$

$$w+x+z=2$$

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 2 \end{bmatrix}$$

$$R_4 - R_1 \rightarrow R_4$$

$$R_2 - R_1 \rightarrow R_2$$

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & -2 & -3 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 \end{bmatrix}$$

$$R_1 - R_3 \rightarrow R_1$$

$$R_2 + 2R_3 \rightarrow R_2$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 \end{bmatrix}$$

$$-2R_2 + R_4 \rightarrow R_4$$

$$R_1 + R_2 \rightarrow R_2$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$0 = -1$$

System is inconsistent.

### Exercise 39

Show that if  $ad - bc \neq 0$ , then the system

$$ax + by = r$$

$$cx + dy = s$$

has a unique solution.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\det(A)} A \begin{bmatrix} r \\ s \end{bmatrix}$$

$$= \frac{1}{ad - bc} A \begin{bmatrix} r \\ s \end{bmatrix}$$

If  $ad - bc \neq 0$ , then the system has a unique solution.

**Exercise 40**

For what values of  $k$ , if any, will the systems have no solution, a unique solution, and infinitely many solutions?

$$\begin{aligned} kx + 2y &= 3 \\ 2x - 4y &= -6 \end{aligned}$$

$$\begin{aligned} A &= \begin{bmatrix} k & 2 & 3 \\ 2 & -4 & -6 \end{bmatrix} \\ &= \begin{bmatrix} k & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} k & 2 & 3 \\ k+1 & 0 & 0 \end{bmatrix} \end{aligned}$$

$k = -1$  will have infinitely many solutions. The system will have a unique solution for all other values.

**Exercise 41**

For what values of  $k$ , if any, will the systems have no solution, a unique solution, and infinitely many solutions?

$$\begin{aligned} x + ky &= 1 \\ kx + y &= 1 \end{aligned}$$

$$A = \begin{bmatrix} 1 & k & 1 \\ k & 1 & 1 \end{bmatrix}$$

$k = 1$  will have infinitely many solutions. The system will have a unique solution for all other values.

**Exercise 42**

For what values of  $k$ , if any, will the systems have no solution, a unique solution, and infinitely many solutions?

$$\begin{aligned} x - 2y + 3z &= 2 \\ x + y + z &= k \\ 2x - y + 4z &= k^2 \end{aligned}$$

$$\begin{aligned} A &= \begin{bmatrix} 1 & -2 & 3 & 2 \\ 1 & 1 & 1 & k \\ 2 & -1 & 4 & k^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & -3 & 2 & 2-k \\ 0 & -3 & 2 & 4-k^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & -3 & 2 & 2-k \\ 0 & 0 & 0 & 2-k+(4-k^2) \end{bmatrix} \end{aligned}$$

$$2 - k + 4 - k^2 = 0$$

$$k^2 + k - 6 = 0$$

$$(k+3)(k-2) = 0$$

$$k = -3 \quad k = 2$$

The system has infinitely many solutions when  $k = 2$ , no solutions otherwise.



**Exercise 43**

For what values of  $k$ , if any, will the systems have no solution, a unique solution, and infinitely many solutions?

$$x + y + kz = 1$$

$$x + ky + z = 1$$

$$kz + y + z = -2$$

$$A = \begin{bmatrix} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & -2 \end{bmatrix}$$

The system has no solution when  $k = 1$ , and a unique solution otherwise.

**Exercise 44**

Give examples of homogeneous systems of  $m$  linear equations in  $n$  variables with  $m = n$  and with  $m > n$  that have infinitely many solutions and a unique solution.

$m = n$ : infinite solutions

$$x + y = 0$$

$$2x + 2y = 0$$

$m = n$ : unique solution

$$x + 2y = 0$$

$$4x + y = 0$$

$m > n$ : infinite solutions

$$x + y = 0$$

$$2x + 2y = 0$$

$$10x + 10y = 0$$

$m > n$ : unique solution

$$x + y = 0$$

$$3x + 1y = 0$$

$$10x + 3y = 0$$

**Exercise 45**

Find the line of intersection of the given planes.

$$3x + 2y + z = -1$$

$$2x - y + 4z = 5$$

Let  $z = t$ :

$$3x + 2y = -1 - t$$

$$2x - y = 5 - 4t$$

$$A = \begin{bmatrix} 3 & 2 & -1 - t \\ 2 & -1 & 5 - 4t \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 4 & -2 - 2t \\ 3 & -1 & 5 - 4t \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 6 & 6t - 12 \\ 3 & -1 & 5 - 4t \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & t - 2 \\ 3 & 0 & 3 - 3t \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & t - 2 \\ 1 & 0 & 1 - t \end{bmatrix}$$

$$l(t) = \begin{cases} x & = t - 2 \\ y & = 1 - t \\ z & = t \end{cases}$$

If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)