Linear Algebra: Homework 3

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Section 1.2

Exercise 48

Find all values of the scalar k for which the two vectors are orthogonal.

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} k+1 \\ k-1 \end{bmatrix}$$

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 0 \\ 2(k+1) + 3(k-1) &= 0 \\ 2k + 2 + 3k - 3 &= 0 \\ 5k &= 1 \\ k &= \frac{1}{5} \end{aligned}$$

Exercise 49

Find all values of the scalar k for which the two vectors are orthogonal.

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} k^2 \\ k \\ -3 \end{bmatrix}$$

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 0 \\ k^2 - k - 6 &= 0 \\ (k-3)(k+2) &= 0 \\ k &= 3 \quad k = -2 \end{aligned}$$

Exercise 50

Describe all vectors that are orthogonal to $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$$3x + y = 0$$

They are all parallel to the line described by $y = -3x$.

Exercise 51

Describe all vectors that are orthogonal to $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$.
They are all parallel to the line described by $ax + by = 0$.

Exercise 52

Under what conditions are the following true for vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ or R^3?

1. $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$: This is true when the vectors are parallel.
2. $\|\vec{u} + \vec{v}\| = \|\vec{u}\| - \|\vec{v}\|$: This is true when the vectors are antiparallel.

Exercise 53

Prove Theorem 1.2(b).

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

Proof:

$$\begin{aligned} \vec{u} \cdot (\vec{v} + \vec{w}) &= \sum_{i=1}^{n} u_i(v_i + w_i) \\ &= \sum_{i=1}^{n} (u_i v_i + u_i w_i) \\ &= \sum_{i=1}^{n} u_i v_i + \sum_{i=1}^{n} u_i w_i \\ &= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \end{aligned}$$

Exercise 54

Prove Theorem 1.2(d).

$$\vec{u} \cdot \vec{u} \geq 0$$
$$\vec{u} \cdot \vec{u} = 0 \quad \text{iff} \quad \vec{u} = \vec{0}$$

Proof:

$$\begin{aligned} \vec{u} \cdot \vec{u} &= \sum_{i=1}^{n} u_i u_i \\ &= \sum_{i=1}^{n} (u_i)^2 \end{aligned}$$

$(u_i)^2$ is non-negative, therefore the summation must be greater than or equal to 0, and only equal to 0 when $\vec{u} = \vec{0}$.

Exercise 55

Prove the stated property of distance between vectors.

$$d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$$

Proof:

$$
\begin{aligned}
d(\vec{u},\vec{v}) &= \sqrt{\sum_{i=1}^{n}(u_i - v_i)^2} \\
&= \sqrt{\sum_{i=1}^{n}(-1)^2(v_i - u_i)^2} \\
&= \sqrt{\sum_{i=1}^{n}(v_i - u_i)^2} \\
&= d(\vec{v},\vec{u})
\end{aligned}
$$

Exercise 56

Prove the stated property of distance between vectors.

$$d(\vec{u},\vec{w}) \leq d(\vec{u},\vec{v}) + d(\vec{v},\vec{w})$$

Proof:

$$
\begin{aligned}
d(\vec{u},\vec{w}) &\leq d(\vec{u},\vec{v}) + d(\vec{v},\vec{w}) \\
\|\vec{u}-\vec{w}\| &\leq \|\vec{u}-\vec{v}\| + \|\vec{v}-\vec{w}\| \\
\|\vec{u}-\vec{w}\| &\leq \|\vec{u}-\vec{v}+\vec{v}-\vec{w}\| \quad \text{(By the triangle inequality)} \\
\|\vec{u}-\vec{w}\| &\leq \|\vec{u}-\vec{w}\|
\end{aligned}
$$

Exercise 57

Prove the stated property of distance between vectors.

$$d(\vec{u},\vec{v}) = 0 \quad \text{iff} \quad \vec{u} = \vec{v}$$

Proof:

$$
\begin{aligned}
0 &= d(\vec{u},\vec{v}) \\
0 &= \sqrt{\sum_{i=1}^{n}(u_i - v_i)^2} \\
0 &= \sum_{i=1}^{n}(u_i - v_i)^2 \\
0 &= \sum_{i=1}^{n}(u_i - v_i) \\
0 &= \sum_{i=1}^{n} u_i - \sum_{i=1}^{n} v_i \\
0 &= \vec{u} - \vec{v} \\
\vec{v} &= \vec{u}
\end{aligned}
$$

Exercise 58

Prove that $\vec{u} \cdot c\vec{v} = c(\vec{u} \cdot \vec{v})$.

$$\begin{aligned}
\vec{u} \cdot c\vec{v} &= \sum_{i=1}^{n} u_i(cv_i) \\
&= \sum_{i=1}^{n} c(u_i v_i) \\
&= c\sum_{i=1}^{n} u_i v_i \\
&= c(\vec{u} \cdot \vec{v})
\end{aligned}$$

Exercise 59

Prove that $\|\vec{u} - \vec{v}\| \geq \|\vec{u}\| - \|\vec{v}\|$.

$$\begin{aligned}
\|\vec{w} + \vec{v}\| &\leq \|\vec{w}\| + \|\vec{v}\| \quad \text{(Triangle Inequality)} \\
Let: \vec{w} &= \vec{u} - \vec{v} \\
\|\vec{u} - \vec{v} + \vec{v}\| &\leq \|\vec{u} - \vec{v}\| + \|\vec{v}\| \\
\|\vec{u}\| &\leq \|\vec{u} - \vec{v}\| + \|\vec{v}\| \\
\|\vec{u}\| - \|\vec{v}\| &\leq \|\vec{u} - \vec{v}\| \\
\|\vec{u} - \vec{v}\| &\geq \|\vec{u}\| - \|\vec{v}\|
\end{aligned}$$

Exercise 60

Suppose know that $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$. Does it follow that $\vec{v} = \vec{w}$? If it does, give a proof that is valid in $\mathbb{R}^n$. Otherwise, give a counterexample.
Suppose $\vec{u} = \vec{0}$. $\vec{v}$ and $\vec{w}$ can be any vector in $\mathbb{R}^n$.

$$\vec{0} \cdot \langle 1, 2 \rangle = \vec{0} \cdot \langle 3, 4 \rangle$$

Exercise 61

Prove that $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = \|\vec{u}\|^2 - \|\vec{v}\|^2$.

$$\begin{aligned}
(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) &= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{v} \\
&= \vec{u} \cdot \vec{u} - \vec{v} \cdot \vec{v} \\
&= \|\vec{u}\|^2 - \|\vec{v}\|^2
\end{aligned}$$

Exercise 62a

Prove that $\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$.

$$\begin{aligned}
\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 &= (\|\vec{u}\|^2 + \|\vec{v}\|^2) + (\|\vec{u}\|^2 + \|\vec{-v}\|^2) \quad \text{(Pythagorean Theorem)} \\
&= \|\vec{u}\|^2 + \|\vec{v}\|^2 + \|\vec{u}\|^2 + \|\vec{v}\|^2 \\
&= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2
\end{aligned}$$

Exercise 63

Prove that $\vec{u} \cdot \vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2$.

$$\begin{aligned}
\frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2 &= \frac{1}{4}\sqrt{(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})}^2 - \frac{1}{4}\sqrt{(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})}^2 \\
&= \frac{1}{4}(\vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} - (\vec{u}\cdot\vec{u} - 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v})) \\
&= \vec{u}\cdot\vec{v}
\end{aligned}$$

Exercise 64a

Prove that $\|\vec{u}+\vec{v}\| = \|\vec{u}-\vec{v}\|$ if and only if $\vec{u}$ and $\vec{v}$ are orthogonal.

$$\begin{aligned}
\|\vec{u}+\vec{v}\| &= \|\vec{u}-\vec{v}\| \\
\sqrt{(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})} &= \sqrt{(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})} \\
(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) &= (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) \\
\vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} &= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} \\
2\vec{u}\cdot\vec{v} &= 0 \\
\vec{u}\cdot\vec{v} &= 0
\end{aligned}$$

The dot product of two vectors is 0 if and only if the two vectors are orthogonal.

Exercise 65a

Prove that $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are orthogonal in $\mathbb{R}$ if and only if $\|\vec{u}\| = \|\vec{v}\|$.

$$\begin{aligned}
(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) &= 0 \\
\vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{v} &= 0 \\
\vec{u}\cdot\vec{u} &= \vec{v}\cdot\vec{v} \\
\sqrt{\vec{u}\cdot\vec{u}} &= \sqrt{\vec{v}\cdot\vec{v}} \\
\|\vec{u}\| &= \|\vec{v}\|
\end{aligned}$$

Exercise 66

If $\|\vec{u}\| = 2, \|\vec{v}\| = \sqrt{3}$, and $\vec{u}\cdot\vec{v} = 1$, find $\|\vec{u}+\vec{v}\|$.

$$\begin{aligned}
\|\vec{u}+\vec{v}\| &= \sqrt{(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})} \\
&= \sqrt{\vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}} \\
&= \sqrt{\|\vec{u}\|^2 + 2(1) + \|\vec{v}\|^2} \\
&= \sqrt{2^2 + 2 + \sqrt{3}^2} \\
&= \sqrt{4+2+3} \\
&= \sqrt{9} = 3
\end{aligned}$$

Exercise 67

Show that there are no vectors $\vec{u}$ and $\vec{v}$ such that $\|\vec{u}\| = 1$, $\|\vec{v}\| = 2$, and $\vec{u} \cdot \vec{v} = 3$.

$$\begin{aligned}
\|\vec{u} + \vec{v}\| &\leq \|\vec{u}\| + \|\vec{v}\| \\
\sqrt{\|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2} &\leq \|\vec{u}\| + \|\vec{v}\| \\
\sqrt{1^2 + 2(3) + 2^2} &\leq 1 + 2 \\
\sqrt{1 + 6 + 4} &\leq 3 \\
\sqrt{11} &\leq 3 \\
3.316 &\leq 3
\end{aligned}$$

Since this case violates the triangle inequality, there can be no such vectors.

Exercise 68

Prove that if $\vec{u}$ is orthogonal to both $\vec{v}$ and $\vec{w}$, then $\vec{u}$ is orthogonal to $\vec{v} + \vec{w}$.

$$\begin{aligned}
\vec{u} \cdot \vec{v} &= 0 \\
\vec{u} \cdot \vec{w} &= 0 \\
\vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} &= 0 \\
\vec{u} \cdot (\vec{v} + \vec{w}) &= 0
\end{aligned}$$

Prove that if $\vec{u}$ is orthogonal to both $\vec{v}$ and $\vec{w}$, then $\vec{u}$ is orthogonal to $s\vec{v} + t\vec{w}$ for all scalars s and t.

$$\begin{aligned}
\vec{u} \cdot \vec{v} &= 0 \\
\vec{u} \cdot \vec{w} &= 0 \\
\vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} &= 0 \\
\vec{u} \cdot (\vec{v} + \vec{w}) &= 0 \\
\sum_{i=1}^{n} u_i(v_i + w_i) &= 0 \\
\sum_{i=1}^{n} u_1 \sum_{i=1}^{n}(v_i + w_i) &= 0
\end{aligned}$$

Exercise 69

Prove that $\vec{u}$ is orthogonal to $\vec{v} - proj_{\vec{u}}\vec{v}$ for all vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, where $\vec{u} \neq \vec{0}$.

$$\begin{aligned}
\vec{u} \cdot (\vec{v} - proj_{\vec{u}}\vec{v}) &= \vec{u} \cdot (\vec{v} - \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\vec{u}) \\
&= \vec{u} \cdot \vec{v} - \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\vec{u} \cdot \vec{u} \\
&= \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} \\
&= 0
\end{aligned}$$

Exercise 70a

Prove that $proj_{\vec{u}}(proj_{\vec{u}}\vec{v}) = proj_{\vec{u}}\vec{v}$.

$$\begin{aligned}
\vec{w} &= proj_{\vec{u}}\vec{v} \\
&= \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u} \\
proj_{\vec{u}}(proj_{\vec{u}}\vec{v}) &= proj_{\vec{u}}\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u} \\
&= proj_{\vec{u}}\vec{w} \\
&= \frac{\vec{u}\cdot\vec{w}}{\vec{u}\cdot\vec{u}}\vec{u} \\
&= \frac{\vec{u}\cdot\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} \\
&= \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\frac{\vec{u}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} \\
&= \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u} \\
&= proj_{\vec{u}}\vec{v}
\end{aligned}$$

Exercise 70b

Prove that $proj_{\vec{u}}(\vec{v} - proj_{\vec{u}}\vec{v}) = 0$.

$$\begin{aligned}
\vec{w} &= proj_{\vec{u}}\vec{v} \\
&= \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u} \\
proj_{\vec{u}}(\vec{v} - proj_{\vec{u}}\vec{v}) &= proj_{\vec{u}}(\vec{v} - \vec{w}) \\
&= \frac{\vec{u}\cdot(\vec{v}-\vec{w})}{\vec{u}\cdot\vec{u}}\vec{u} \\
&= \frac{\vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{w}}{\vec{u}\cdot\vec{u}}\vec{u} \\
&= \frac{\vec{u}\cdot\vec{v} - \vec{u}\cdot\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} \\
&= \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}} - \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\frac{\vec{u}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u} \\
&= \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}} - \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u} \\
&= 0\vec{u} = 0
\end{aligned}$$

Exercise 73

Use the fact that $proj_{\vec{u}}\vec{v} = c\vec{u}$ for some scalar c together with Figure 1.41 to find c and derive the formula for $proj_{\vec{u}}\vec{v}$.

$$\begin{aligned}
c\vec{u} \cdot (\vec{v} - c\vec{u}) &= 0 \\
(c\vec{u} \cdot \vec{v}) - (c\vec{u} \cdot c\vec{u}) &= 0 \\
c(\vec{u} \cdot \vec{v}) &= |c|^2(\vec{u} \cdot \vec{u}) \\
\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} &= |c| \\
proj_{\vec{u}}\vec{v} = c\vec{u} &= \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\vec{u}
\end{aligned}$$

Exercise 74

Using mathematical induction, prove the following generalization of the Triangle Inequality:

$$\|\vec{v_1} + \vec{v_2} + \cdots + \vec{v_n}\| \leq \|\vec{v_1}\| + \|\vec{v_2}\| + \cdots + \|\vec{v_n}\|$$

Basis:

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Assumption:

$$\|\vec{v_1} + \vec{v_2} + \cdots + \vec{v_n}\| \leq \|\vec{v_1}\| + \|\vec{v_2}\| + \cdots + \|\vec{v_n}\|$$

Induction:

$$\begin{aligned}
\|\vec{v_1} + \vec{v_2} + \cdots + \vec{v_n} + \vec{v_{n+1}}\| &\leq \|\vec{v_1} + \vec{v_2} + \cdots + \vec{v_n}\| + \|\vec{v_{n+1}}\| \\
&\leq \|\vec{v_1}\| + \|\vec{v_2}\| + \cdots + \|\vec{v_n}\| + \|\vec{v_{n+1}}\|
\end{aligned}$$

Section 1.3

Exercise 1

Write the equation of the line passing through P with normal vector $\vec{n}$ in normal form and general form.

$$P = (0,0) \quad \vec{n} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$
$$3x + 2y = 0$$

Exercise 3

Write the equation of the line passing through P with direction vector $\vec{d}$ in vector form and parametric form.

$$P = (1,0) \quad \vec{d} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$l = \begin{cases} x &= 1 - t \\ y &= 3t \end{cases}$$

Exercise 5

Write the equation of the line passing through P with direction vector $\vec{d}$ in vector form and parametric form.

$$P = (0,1,0) \quad \vec{d} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$l = \begin{cases} x & = t \\ y & = 1 - t \\ z & = 4t \end{cases}$$

Exercise 7

Write the equation of the line passing through P with normal vector $\vec{n}$ in normal form and general form.

$$P = (0,1,0) \quad \vec{n} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$3x + 2y + z = 2$$

Exercise 9

Write the equation of the plane passing through P with vectors $\vec{u}$ and $\vec{v}$ in vector form and parametric form.

$$P = (0,0,0) \quad \vec{u} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$$\Pi = \begin{cases} x & = 2s - t3 \\ y & = s + 2t \\ z & = 2s + t \end{cases}$$

Exercise 11

Give the vector equation of the line passing through P and Q.

$$P = (1,-2) \quad Q = (3,0)$$

$$\vec{d} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Exercise 13

Give the vector equation of the plane passing through P, Q, R.

$$P = (1, 1, 1) \quad Q = (4, 0, 2) \quad R = (0, 1, -1)$$

$$\vec{u} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$
$$\vec{v} = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$$

Exercise 15

Find parametric equations and an equation in vector form for the lines in $\mathbb{R}^2$ with the following equations:

$$y = 3x - 1$$

$$l = \begin{cases} x & = t \\ y & = 3t - 1 \end{cases}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Exercise 17

Suggest a "vector proof" of the fact that in $\mathbb{R}^2$, two lines with slope m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$.

$$\begin{aligned} \vec{u} &= \langle m_1, 1 \rangle \\ \vec{v} &= \langle m_2, 1 \rangle \\ \vec{u} \cdot \vec{v} &= 0 \\ m_1 m_2 + 1 &= 0 \\ m_1 m_2 &= -1 \end{aligned}$$

Exercise 19

The plane P_1 has the equation $4x - y + 5z = 2$. For each of the planes P in Exercise 18. determine whether P_1 and P are parallel, perpendicular, or neither.

$$\begin{aligned}
\vec{n_1} &= \langle 4, -1, 5 \rangle \\
a: 2x + 3y - z &= 1 \\
\vec{n_a} &= \langle 2, 3, -1 \rangle \\
\vec{n_1} &\neq c\vec{n_a} \\
\vec{n_1} \cdot \vec{n_a} &= 8 + (-3) + (-5) = 0 && \text{Perpendicular} \\
b: 4x - y + 5z &= 0 \\
\vec{n_b} &= \langle 4, -1, 5 \rangle \\
\vec{n_1} &= c\vec{n_a} && \text{Parallel} \\
c: x - y - z &= 3 \\
\vec{n_c} &= \langle 1, -1, -1 \rangle \\
\vec{n_1} &\neq c\vec{n_c} \\
\vec{n_1} \cdot \vec{n_c} &= 4 + 1 + (-5) = 0 && \text{Perpendicular} \\
d: 4x + 6y - 2z &= 0 \\
\vec{n_d} &= \langle 4, 6, -2 \rangle = \langle 2, 3, -1 \rangle \\
\vec{n_1} &\neq c\vec{n_d} \\
\vec{n_1} \cdot \vec{n_d} &= 8 + (-3) + (-5) = 0 && \text{Perpendicular}
\end{aligned}$$

Exercise 21

Find the vector form of the equation of the line in $\mathbb{R}^2$ that passes through $P = (2, -1)$ and is parallel to the line with general equation $2x - 3y = 1$.

$$\vec{d} = \langle 2, 3 \rangle$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Exercise 23

Find the vector form of the equation of the line in $\mathbb{R}^3$ that passes through $P = (-1, 0, 3)$ and is parallel to the line with parametric equations:

$$l = \begin{cases} x &= 1 - t \\ y &= 2 + 3t \\ z &= -2 - t \end{cases}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$

Section 2.1

Exercise 1

Determine which equations are linear in the variables x, y, z.

$$x - \pi y + \sqrt[3]{5}z = 0$$

Linear.

Exercise 3

Determine which equations are linear in the variables x, y, z.

$$x^{-1} + 7y + z = \sin \frac{\pi}{9}$$

Nonlinear, x is not 1st degree.

Exercise 5

$$3\cos(x) - 4y + z = \sqrt{3}$$

Nonlinear, $\cos(x)$ is a function of x.

Exercise 11

Find the solution set of each equation.

$$3x - 6y = 0$$

All points that lie on the line $y = \frac{1}{2}x$.

Exercise 13

Find the solution set of each equation.

$$x + 2y + 3z = 4$$

All points that lie on the plane $x + 2y + 3z = 4$ or $[4 - 3t - 2s, s, t]$.

Exercise 15

Determine geometrically whether each system has a unique solution, infinitely many solutions, or no solution. Then solve each system algebraically to confirm your answer.

$$x + y = 0 \quad 2x + y = 3$$

Two intersecting lines, one solution at (3,-3).

Exercise 17

Determine geometrically whether each system has a unique solution, infinitely many solutions, or no solution. Then solve each system algebraically to confirm your answer.

$$3x - 6y = 3 \quad -x + 2y = 1$$

Two parallel lines, no solution.

Exercise 19

Solve the given system by back substitution.

$$\begin{aligned} x - 2y &= 1 \\ y &= 3 \\ x &= 7 \end{aligned}$$

Exercise 21

Solve the given system by back substitution.

$$\begin{aligned} x - y + z &= 0 \\ 2y - z &= 1 \\ 3z &= -1 \\ z &= -\frac{1}{3} \\ y &= \frac{1}{3} \\ x &= \frac{2}{3} \end{aligned}$$

Exercise 23

Solve the given system by back substitution.

$$\begin{aligned} x_1 + x_2 - x_3 - x_4 &= 1 \\ x_2 + x_3 + x_4 &= 0 \\ x_3 - x_4 &= 0 \\ x_4 &= 1 \\ x_3 &= 1 \\ x_2 &= -2 \\ x_1 &= 1 \end{aligned}$$

Exercise 25

Solve these systems.

$$\begin{aligned} x &= 2 \\ 2x + y &= -3 \\ -3x - 4y + z &= -1 \\ y &= -7 \\ z &= -23 \end{aligned}$$

Exercise 27

Find the augmented matrices of the linear systems.

$$\begin{aligned} x - y &= 0 \\ 2x + y &= 3 \\ A &= \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 1 & 3 \end{array}\right] \end{aligned}$$

Exercise 29

Find the augmented matrices of the linear systems.

$$\begin{aligned} x + 5y &= -1 \\ -x + y &= -5 \\ 2x + 4y &= 4 \end{aligned}$$

$$A = \left[\begin{array}{cc|c} 1 & 5 & -1 \\ -1 & 1 & -5 \\ 2 & 4 & 4 \end{array}\right]$$

Exercise 31

Find a system of linear equations that has the given matrix as its augmented matrix.

$$A = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 1 \end{array}\right]$$

$$\begin{aligned} y + z &= 1 \\ x - y &= 1 \\ 2x - y + z &= 1 \end{aligned}$$

Exercise 33

Solve the linear systems in the given equations.

$$\begin{aligned} x - y &= 0 \\ 2x + y &= 3 \end{aligned}$$

$$A = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 1 & 3 \end{array}\right]$$

$$-2R_1 + R_2 \to R_2$$

$$A = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 3 & 3 \end{array}\right]$$

$$\frac{1}{3}R_2 \to R_2$$

$$A = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

$$R_2 + R_1 \to R_1$$

$$A = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

$$\begin{aligned} x &= 1 \\ y &= 1 \end{aligned}$$

Exercise 35

Solve the linear systems in the given equations.

$$x + 5y = -1$$
$$-x + y = -5$$
$$2x + 4y = 4$$
$$A = \left[\begin{array}{cc|c} 1 & 5 & -1 \\ -1 & 1 & -5 \\ 2 & 4 & 4 \end{array}\right]$$
$$R_1 + R_2 \to R_2$$
$$-2R_1 + R_3 \to R_3$$
$$A = \left[\begin{array}{cc|c} 1 & 5 & -1 \\ 0 & 6 & -6 \\ 0 & -6 & 6 \end{array}\right]$$
$$\frac{1}{6}R_2 \to R_2$$
$$\frac{1}{6}R_3 \to R_3$$
$$A = \left[\begin{array}{cc|c} 1 & 5 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{array}\right]$$
$$5R_3 + R_1 \to R_1$$
$$A = \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{array}\right]$$
$$x = 4$$
$$y = -1$$

Exercise 37

Solve the linear systems in the given equations.

$$A = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 1 \end{array}\right]$$

$$R_2 \leftrightarrow R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 2 & -1 & 1 & 1 \end{array}\right]$$

$$-2R_1 + R_3 \rightarrow R_3$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$R_3 - R_2 \rightarrow R_3$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$$0 = 2 \quad \text{(Inconsistent)}$$

Exercise 39

Find a system of two linear equations in the variables x and y whose solution set is given by the parametric equations $x = t$ and $y = 3 - 2t$.

$$y = 3 - 2x$$

$$3x = 15$$

Find another parametric solution to the system in the first part in which the parameter is s and $y = s$.

$$y = s$$

$$x = \frac{s-3}{-2}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech