

Linear Algebra

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Similar Matrices and Diagonalization

Throughout A, B will be $n \times n$ matrices unless stated otherwise. We say A is **similar to** B if there exists $n \times n$ matrix P such that $P^{-1}AP = B$. We denote this as $A \sim B$. Suppose we have:

$$P^{-1}AP = D$$

To form P , the columns of P are the eigenvectors of A . To form D , put the eigenvalues of A on the main diagonal of D . If the eigenvector \vec{v} is in the column i of A , then its corresponding eigenvalue goes in column i of D .

Theorems and Lemmas

Theorem 1: A, B, C are $n \times n$ matrices, then:

(i) $A \sim A$

$$I^{-1}AI = A$$

(ii) $A \sim B \rightarrow B \sim A$

$$A = PBP^{-1}$$

$$\text{Let } Q = P^{-1}$$

$$A = Q^{-1}BQ$$

$$\therefore B \sim A$$

(iii) $A \sim B \wedge B \sim C \rightarrow A \sim C$

$$\begin{aligned}
 A \sim B &\rightarrow P^{-1}AP = B \\
 B \sim C &\rightarrow Q^{-1}BQ = C \\
 B &= QCQ^{-1} \\
 P^{-1}AP &= QCQ^{-1} \\
 Q^{-1}P^{-1}APQ &= C \\
 (PQ^{-1})A(PQ) &= C \\
 \text{Let } E &= PQ \\
 E^{-1}AE &= C \\
 \therefore A &\sim C
 \end{aligned}$$

Theorem 2: Suppose $A \sim B$, then:

(a) $|A| = |B|$

$$\begin{aligned}
 P^{-1}AP &= B \\
 |P^{-1}AP| &= |B| \\
 |P^{-1}||A||P| &= |B| \\
 \frac{1}{|P|}|A||P| &= |B| \\
 |A| &= |B|
 \end{aligned}$$

(b) A invertible $\leftrightarrow B$ invertible

(c) A, B have the same rank.

(d) A, B have the same characteristic polynomial.

$$\begin{aligned}
 \text{characteristic polynomial of } B &= |B - \lambda I| \\
 &= |P^{-1}AP - \lambda I| \\
 &= |P^{-1}AP - P^{-1}(\lambda I)P| \\
 &= |P^{-1}(A - \lambda I)P| = |P^{-1}||A - \lambda I||P| \\
 &= \frac{1}{|P|}|A - \lambda I||P| \\
 &= |A - \lambda I| \quad \text{characteristic polynomial of } A
 \end{aligned}$$

(e) A, B have the same eigenvalues. Eigenvalues are the roots of the characteristic polynomial, thus this follows from part (d).

(f) $A^m \sim B^m$ for all integers $m \geq 0$

$$\begin{aligned}(P^{-1}AP)^2 &= (P^{-1}AP)(P^{-1}AP) \\ &= P^{-1}A^2P \\ (P^{-1}AP)^2 &= (P^{-1}A^2P)(P^{-1}AP) \\ &= P^{-1}A^3P\end{aligned}$$

and this can be proven through induction.

(g) If A is invertible, then $A^m \sim B^m$ for all integers m .

Theorem 3: We say A is **diagonalizable** if there exists an $n \times n$ diagonal matrix D such that $A \sim D$. A is diagonalizable if and only if A has n linearly independent eigenvectors.

$$\begin{aligned}P^{-1}AP &= D \\ AP &= PD \\ P &= [\vec{p}_1 \mid \vec{p}_2 \mid \cdots \mid \vec{p}_m] \\ AP &= A[\vec{p}_1 \mid \vec{p}_2 \mid \cdots \mid \vec{p}_m] \\ &= [A\vec{p}_1 \mid A\vec{p}_2 \mid \cdots \mid A\vec{p}_m] \\ PD &= P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \\ A\vec{p}_1 &= \lambda\vec{p}_1 \\ A\vec{p}_2 &= \lambda\vec{p}_2 \\ &\vdots \\ A\vec{p}_n &= \lambda\vec{p}_n\end{aligned}$$

The vectors \vec{p}_i are eigenvectors of A with eigenvalues λ_i .

Theorem 4: Let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of A . If B_i is the basis of E_{λ_i} , then

$\bigcup_i B_i$ is linearly independent.

$$B_i = \{v_{i1}, v_{i2}, \dots, v_{in_i}\}$$

$$\sum_{i=1}^k \sum_{j=1}^{n_i} c_{ij} v_{ij} = \vec{0}$$

$$\text{Let } \vec{x}_i = \sum_{j=1}^{n_i} c_{ij} v_{ij}$$

$$\vec{x}_1 + \vec{x}_2 + \dots + \vec{x}_k = \vec{0}$$

$$\vec{x}_i \in E_{\lambda_i}$$

$$\text{All } \vec{x}_i = 0$$

$$\text{All } c_{ij} = 0$$

$\therefore B$ is linearly independent

Theorem 5: The **algebraic multiplicity** of eigenvalue λ_i is the number of times it is an eigenvalue of A and the **geometric multiplicity** is the dimension of the eigenspace E_{λ_i} . The geometric multiplicity of each eigenvalue is less than or equal to its algebraic multiplicity.

The Diagonalization Theorem

Say A has distinct eigenvalues $\lambda_1, \dots, \lambda_k$. The following statements are equivalent:

- (a) A is diagonalizable.
- (b) The union of the bases of the eigenspaces of A contains exactly n vectors.
- (c) The algebraic multiplicity of each eigenvalue equals its geometric multiplicity

Diagonalization Procedure

- 1) Find the characteristic polynomial $p(\lambda) = |A - \lambda I|$.
- 2) Find all the roots of $p(\lambda)$. Those are the eigenvalues.
- 3) Compute a basis for each eigenspace. If $\dim(E_{\lambda_i}) <$ algebraic multiplicity for λ_i for any i , stop. In this case, we can't diagonalize A .
- 4) Form P and D such that $P^{-1}AP = D$

Example

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

Diagonalize A if possible. If it is not diagonalizable, explain why not. Find the roots of $|A - \lambda I| = 0$, we get eigenvalues:

$$\lambda_1 = \lambda_2 = 1, \lambda_3 = 2$$

For $\lambda_1 = \lambda_2 = 1$:

$$E_1 = \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$
$$E_2 = \text{span} \left(\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right)$$

A is not diagonalizable because λ_1 has algebraic multiplicity 2 but $\dim(E_{\lambda_1}) = 1$.

Example

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

Find the roots of $|A - \lambda I|$, we get eigenvalues:

$$\lambda_1 = \lambda_2 = 0, \lambda_3 = -2$$

$$E_0 = \text{span} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$
$$E_{-2} = \text{span} \left(\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right)$$

Thus A is diagonalizable.

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$
$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech