Linear Algebra

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Linear Transformations

$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a transformation if for each $\vec{x} \in \mathbb{R}^n$ $\exists!$ $T(\vec{x}) \in \mathbb{R}^m$. $Range(T) = Image(T) = \{T(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\}$. $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is one-to-one if $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$,

$$T(\vec{x}) = T(\vec{y}) \longrightarrow \vec{x} = \vec{u}$$

$$\vec{x} \neq \vec{y} \longrightarrow T(\vec{x}) \neq \vec{y}$$

$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **onto** (surjective) if $Range(T) = Codomain(T)$.

Properties of Linear Transformations

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a transformation. T is a linear transformation if for all $\vec{u}, \vec{v} \in \mathbb{R}^n$ and all scalars c, d:

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
2. $T(c\vec{u}) = cT(\vec{u})$
3. $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{u})$

Example

Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3y \\ -4x \end{bmatrix}$$

Verify that T is linear. Let:

$$\vec{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\begin{aligned}
T(\vec{u} + \vec{v}) &= T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) \\
&= \begin{bmatrix} 3(y_1 + y_2) \\ -4(x_1 + x_2) \end{bmatrix} \\
&= \begin{bmatrix} 3y_1 + 3y_2 \\ -4x_1 - 4x_2 \end{bmatrix} \\
&= \begin{bmatrix} 3y_1 \\ -4x_1 \end{bmatrix} + \begin{bmatrix} 3y_2 \\ -4x_2 \end{bmatrix} \\
&= T(\vec{u}) + T(\vec{v}) \\
T(c\vec{u}) &= \left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) \\
&= \begin{bmatrix} 3(cy) \\ -4(cx) \end{bmatrix} \\
&= c\begin{bmatrix} 3y \\ -4x \end{bmatrix} \\
&= cT(\vec{u})
\end{aligned}$$

T satisfies both properties of a linear transformation. To prove a transformation is not linear, one only needs to find a single counterexample.

Example

Consider $T : \mathbb{R}^n \to \mathbb{R}^m$ as a linear transformation. What is $T(\vec{0})$?

$$\begin{aligned}
T(\vec{0}) &= T(\vec{0} + \vec{0}) \\
&= T(\vec{0}) + T(\vec{0}) \\
\vec{0} &= T(\vec{0})
\end{aligned}$$

Facts about Linear Transformations

Suppose $B = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ basis for R^n. Let $\vec{x} \in \mathbb{R}^n$.

$$\begin{aligned} \vec{x} &= \sum_{i=1}^{n} x_1 \vec{v_1} \\ T(\vec{x}) &= T\left(\sum_{i=1}^{n} x_i \vec{v_i}\right) \\ &= \sum_{i=1}^{n} x_i T(\vec{v_i}) \end{aligned}$$

Let the standard matrix A_T stand for the linear transformation T:

$$\begin{aligned} A_T &= [A] \\ &= \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & \ldots & T(\vec{e_n}) \end{bmatrix} \end{aligned}$$

Let $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^k$. The composition of linear transformations $T \circ S : \mathbb{R}^n \to \mathbb{R}^k$ is also linear.

$$\begin{aligned} [T] &= B \\ [S] &= A \\ (T \circ S)(\vec{x}) &= T(S(\vec{x})) \\ &= T(A\vec{x}) \\ &= B(A\vec{x}) \\ &= (BA)\vec{x} \end{aligned}$$

Example

Define $\Pi_x : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\Pi_x\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

Π_x is linear. Find $[\Pi_x]$:

$$\begin{aligned}
\Pi_x(\vec{e_1}) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\Pi_x(\vec{e_2}) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
A = [\Pi_x] &= \begin{bmatrix} \Pi_x(\vec{e_1}) & \Pi_x(\vec{e_2}) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech