Linear Algebra

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Subspaces

Consider $\mathbb{R}^n$. Let $S \subseteq \mathbb{R}^n$. S is a **subspace** of $\mathbb{R}^n$ if:

1. $\vec{0} \in S$
2. If $\vec{u}, \vec{v} \in S$, then $\vec{u} + \vec{v} \in S$. (Additive Closure)
3. If $\vec{u} \in S$ and c is a scalar, $c\vec{u} \in S$. (Closure over Scalar Multiple)

Consider $S = \{\vec{0}\}$. Is S a subspace of $\mathbb{R}^n$?

1. $\vec{0} \in S$
2. $\vec{0} + \vec{0} = \vec{0} \quad \vec{0} \in S$
3. $c\vec{0} = \vec{0} \quad \vec{0} \in S$

$\therefore S = \{\vec{0}\}$ is a subspace of $\mathbb{R}^n$.

Example

Let P be a plane through the origin. Show P is a subspace of $\mathbb{R}^3$.

$$P = span(\vec{v}, \vec{w})$$

1. $0\vec{v} + 0\vec{w} = \vec{0} \in P$
2. Additive Closure:

$$\vec{\alpha_1} = c_1\vec{v} + c_2\vec{w} \in P$$
$$\vec{\alpha_2} = d_1\vec{v} + d_2\vec{w} \in P$$
$$\vec{\alpha_1} + \vec{\alpha_2} = (c_1 + d_1)\vec{v} + (c_2 + d_2)\vec{w} \in P$$

3. Closure over Scalar Multiple:

$$\begin{aligned}\vec{\alpha} &= c_1\vec{v} + c_2\vec{w} \in P \\ c\vec{\alpha} &= c(c_1\vec{v} + c_2\vec{w}) \\ &= (cc_1)\vec{v} + (cc_2)\vec{w} \in P\end{aligned}$$

So $P \subseteq \mathbb{R}^3$

Theorem

In $\mathbb{R}^n$, $span(\vec{v_1}, \dots, \vec{v_m})$ is a subspace of $\mathbb{R}^n$. Non-example, consider:

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = 3y + 1, z = 2y \right\}$$

Is S a subspace of $\mathbb{R}^3$?

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y + 1 \\ y \\ 2y \end{bmatrix}$$

Thus $\vec{0} \notin S$. S is not a subspace of $\mathbb{R}^3$.

Subspaces Associated with Matrices

Let A be an $m \times n$ matrix.

1. $row(A) =$ span of the rows of A
2. $col(A) =$ span of the cols of A
3. $null(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$

Basis/Dimension for a Subspace

A basis $B = \{\vec{v_1}, \dots, \vec{v_k}\}$ for S is a linearly independent set such that $span(B) = S$. The dimension S is the number of elements of basis B. If we have m vectors in a subspace of dimension n, where $m > n$, then this set is linearly dependent. The dimension $dim(S)$ is the number of elements in a basis for S. Any $\vec{x} \in S$ can be written uniquely as a linear combination of elements of B.

Fundamental Theorem of Invertible Matrices

Let A be an $n \times n$ matrix. The following are equivalent:

1. A is invertible.
2. $A\vec{x} = \vec{b}$ has a unique solution $\forall b \in \mathbb{R}^n$.
3. $A\vec{x} = \vec{0}$ has only the trivial solution.
4. The reduced row echelon form of A is I_n.
5. A is the product of elementary matrices.
6. $rank(A) = n$
7. $nullity(A) = 0$
8. The column vectors of A are linearly independent.
9. The column vectors of A span $\mathbb{R}^n$.
10. The column vectors of A form a basis for $\mathbb{R}^n$.
11. The row vectors of A are linearly independent.
12. The row vectors of A span $\mathbb{R}^n$.
13. The row vectors of A form a basis of $\mathbb{R}^n$.

Definition: The rank of a matrix A:

$$rank(A) = dim(col\ A) = dim(row\ A)$$

Definition: $nullity(A) = dim(null(A))$
Definition: Let A be $m \times n$. then

$$n = rank(A) + nullity(A)$$

Example

$$M = \begin{bmatrix} 2 & 3 \\ 1 & 5 \\ 4 & 7 \\ 3 & 6 \end{bmatrix}$$

$$N = \begin{bmatrix} 2 & 1 & -2 & -1 \\ 4 & 4 & -3 & 1 \\ 2 & 7 & 1 & 8 \end{bmatrix}$$

Find the rank and nullity of M and N.

For M the columns are clearly linearly independent, so $rank(M) = 2$, and $nullity(M) = 0$.

For N:

$$N \begin{bmatrix} 2 & 1 & -2 & -1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $rank(N) = 2$ and $nullity(N) = 2$.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech