

Linear Algebra

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Spanning Sets and Linear Independence

We want to know when one vector is a **linear combination** of some other vectors.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$$

For example, is the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ a linear combination of $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$.

The solution would be to find scalars c_1, c_2, c_3 such that:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$$

We can treat this as a system of linear equations and solve an augmented matrix:

$$A' = \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & -3 & 3 \end{array} \right]$$

Bring this to reduced row echelon form yields:

$$A' = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right]$$

Thus, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a linear combination of \vec{v}, \vec{w} through the constants:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Describing a Linear System

There are three equivalent ways to specify a linear system:

1.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = b_1 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = b_m \end{cases}$$

2. Let \vec{A}_i be a column vector with entries a_i :

$$x_1\vec{A}_1 + x_2\vec{A}_2 + \cdots + x_n\vec{A}_n = \vec{b}$$

3. $A\vec{x} = \vec{b}$ where A is an $m \times x$ matrix:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

where we multiply $A\vec{x} = \vec{b}$ using:

$$b_i = \text{ith row of } A \times \vec{x}$$

Theorem: A system of linear equations with augmented matrix $[A|\vec{b}]$ is consistent if and only if \vec{b} is a linear combination of the columns of A .

Spanning Sets

Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors.

$$\text{Span}(\{\vec{v}_1, \dots, \vec{v}_k\}) = \left\{ \sum_{i=1}^k c_i \vec{v}_i \mid c_i \text{'s are scalars} \right\}$$

The span of:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is equal to:

$$\begin{aligned} \text{Span}(\{\vec{e}_1, \vec{e}_2\}) &= \{c_1\vec{e}_1 + c_2\vec{e}_2 \mid c_1, c_2 \in \mathbb{R}\} \\ &= \left\{ \begin{bmatrix} c_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\} \\ &= \mathbb{R}^2 \end{aligned}$$

Is a vector \vec{v} in $\text{Span}(\{\vec{u}, \vec{w}\})$? Can we find scalars c_1, c_2 such that $\vec{v} = c_1\vec{u} + c_2\vec{w}$? This translates to the following:

$$A = [\vec{u} \quad \vec{w}] \quad \vec{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

And we are essentially solving a system of linear equations:

$$A' = [A \mid \vec{v}]$$

Linear Independence

We say vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ are **linearly independent** if:

$$\sum_{i=1}^n c_i \vec{v}_i = \vec{0} \Rightarrow \forall i (c_i = 0)$$

In terms of matrices, let $A = [\vec{v}_1 \quad \vec{v}_2 \quad \dots \quad \vec{v}_m]$. Examining $A\vec{x} = \vec{0}$, the only solution for \vec{x} is:

$$\vec{x} = \begin{bmatrix} c_1 \\ c_2 \\ \dots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}$$

A set of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ is **linearly dependent** if it is not linearly independent.

Example

Are the vectors $\vec{0}, \vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ linearly dependent or independent?

$$c_1(\vec{0}) + c_2\vec{v}_1 + c_3\vec{v}_2 + \dots + c_{m+1}\vec{v}_m = \vec{0}$$

c_1 can be 1, so this set of vectors is linearly dependent.

Example

Are the vectors

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

linearly independent or linearly dependent?

$$c_1\vec{v} + c_2\vec{w} \stackrel{?}{=} \vec{0}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This has a unique solution, so \vec{v}, \vec{w} are linearly independent.

Example

The general equation of the plane contains points $(1,0,3)$, $(-1,1,-3)$, and the origin. It has equation $ax + by + cz = 0$. Find a, b, c .

$$a(1) + 3c = 0$$

$$a(-1) + 1b - 3c = 0$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ -1 & 1 & -3 & 0 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right]$$

$$a + 3c = 0$$

$$a = -3c$$

$$b = 0$$

$$-3cx + cz = 0$$

$$-3x + z = 0$$

Example

Prove that $\vec{u}, \vec{v}, \vec{w} \in \text{span}(\vec{u}, \vec{v}, \vec{w})$.

$$\vec{u} = 1\vec{u} + 0\vec{v} + 0\vec{w}$$

$$\vec{v} = 0\vec{u} + 1\vec{v} + 0\vec{w}$$

$$\vec{w} = 0\vec{u} + 0\vec{v} + 1\vec{w}$$

$$\vec{u}, \vec{v}, \vec{w} \in \text{span}(\vec{u}, \vec{v}, \vec{w})$$

Example

Prove that $\vec{u}, \vec{v}, \vec{w} \in \text{span}(\vec{u}, \vec{u} + \vec{v}, \vec{u} + \vec{v} + \vec{w})$.

$$\vec{u} = 1\vec{u} + 0(\vec{u} + \vec{v}) + 0(\vec{u} + \vec{v} + \vec{w})$$

$$\vec{v} = -1\vec{u} + 1(\vec{u} + \vec{v}) + 0(\vec{u} + \vec{v} + \vec{w})$$

$$\vec{w} = 0\vec{u} - 1(\vec{u} + \vec{v}) + 1(\vec{u} + \vec{v} + \vec{w})$$

$$\vec{u}, \vec{v}, \vec{w} \in \text{span}(\vec{u}, \vec{u} + \vec{v}, \vec{u} + \vec{v} + \vec{w})$$

Useful Fact

Suppose we have m vectors in \mathbb{R}^n , where $m > n$. Those vectors are linearly dependent.

$$A = [\vec{v}_1 \quad \vec{v}_2 \quad \dots \quad \vec{v}_m]$$

Examine: $A\vec{x} = \vec{0}$. As a consequence of the rank theorem, which states that the number of free variables is equal to the number of columns of A minus the rank of A , $A\vec{x} = \vec{0}$ has a non-trivial solution, hence the columns of A are linearly dependent.

Example

Prove that if $\vec{u}_1, \dots, \vec{u}_m \in \mathbb{R}^n$ where $S = \{\vec{u}_1, \dots, \vec{u}_k\}$ and $T = \{\vec{u}_1, \dots, \vec{u}_k, \vec{u}_{k+1}, \dots, \vec{u}_m\}$ that $\text{span}(S) \subseteq \text{span}(T)$. Let $\alpha \in \text{span}(S)$:

$$\begin{aligned} \alpha &= c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_k\vec{u}_k \\ &= c_1\vec{u}_1 + \dots + c_k\vec{u}_k + 0\vec{u}_{k+1} + 0\vec{u}_{k+2} + \dots + 0\vec{u}_m \\ &\in \text{span}(T) \end{aligned}$$

Deduce if $\mathbb{R}^n = \text{span}(S)$, then $\mathbb{R}^n = \text{span}(T)$.

$$\mathbb{R}^n = \text{span}(S) \subseteq \text{span}(T) \subseteq \mathbb{R}^n$$

$$\text{span}(T) = \mathbb{R}^n$$

Example

Suppose \vec{w} is a linear combination of $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$ and each \vec{u}_i is a linear combination of $\vec{v}_1, \dots, \vec{v}_m$. Prove \vec{w} is a linear combination of $\vec{v}_1, \dots, \vec{v}_m$.

$$\begin{aligned}\vec{w} &= \sum_{i=1}^k c_i \vec{u}_i \\ &= \sum_{i=1}^k c_i \left(\sum_{j=1}^m d_{ij} \vec{v}_j \right) \\ &= \sum_{i=1}^k \left(\sum_{j=1}^m c_i d_{ij} \vec{v}_j \right) \\ &= \sum_{j=1}^m \left(\sum_{i=1}^k c_i d_{ij} \right) \vec{v}_j \\ w &\in \text{span}(\vec{v}_1, \dots, \vec{v}_m)\end{aligned}$$

Also suppose each \vec{v}_j is a linear combination of $\vec{u}_1, \dots, \vec{u}_k$. Prove $\text{span}(\vec{u}_1, \dots, \vec{u}_k) = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$. Above we proved:

$$\text{span}(\vec{u}_1, \dots, \vec{u}_k) \subseteq \text{span}(\vec{v}_1, \dots, \vec{v}_m)$$

Therefore:

$$\text{span}(\vec{u}_1, \dots, \vec{u}_k) \supseteq \text{span}(\vec{v}_1, \dots, \vec{v}_m)$$

Example

If the columns of an $n \times n$ matrix A are linearly independent as vectors in \mathbb{R}^n , what is the rank of A ?

$$\text{rank}(A) = n$$

The columns are linearly independent as $A\vec{x} = \vec{0}$ has only the trivial solution. So there are no free variables.

Example

Prove two vectors are linearly dependent if and only if 1 is a scalar multiple of the other.

Say \vec{u} and \vec{v} are linearly dependent, there exists scalars c_1, c_2 (not both 0) such that $c_1\vec{u} + c_2\vec{v} = \vec{0}$. Without loss of generality, we say that $c_1 \neq 0$, then:

$$\begin{aligned}c_1\vec{u} &= -c_2\vec{v} \\ \vec{u} &= \frac{-c_2}{c_1}\vec{v}\end{aligned}$$

Suppose \vec{u} is a scalar multiple of \vec{v} .

$$\begin{aligned}\vec{u} &= c\vec{v} \\ c\vec{v} - \vec{u} &= \vec{0}\end{aligned}$$

Thus, \vec{u} and \vec{v} are linearly dependent.

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech