Linear Algebra

Alvin Lin

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Systems of Linear Equations

A **linear equation** in variables $x_1, x_2, \dots, x_n$ can be written in this form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \dots, a_n, b$ are constants. The following examples are all linear equations:

1. $4x - 5y = -2$
2. $r - \frac{1}{3}s + \frac{1}{5}t = \pi^2$
3. $2x_1 + 4x_2 = 5 - x_3 + 5x_4$
4. $\sqrt{3}x + \frac{\pi}{4}y - \sin(\frac{\pi}{5})z = 1$

The following examples are NOT linear equations:

1. $2xy + 3z = 10$
2. $(x_1)^2 - (x_2)^2 = 47$
3. $\frac{x}{y} + 4z = 936$
4. $\sqrt{2x} + \frac{\pi}{4}y - \sin(\frac{\pi}{12}z) = 2$
5. $\sin(x_1) + 4x_2 + 4^{x_3} = 22$

Example

Consider $3x - 4y = -1$. Characterize all solutions to this linear system. Let $x = t$.

$$\begin{aligned} 3t - 4y &= 1 \\ -4y &= -1 - 3t \\ y &= \frac{1}{4} + \frac{3}{4}t \\ \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ \frac{1}{4} + \frac{3}{4}t \end{bmatrix} &= \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} + t \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} \end{aligned}$$

Solving Systems of Linear Equations

A system of **linear equations** is a finite collection of linear equations. Solving the system involves finding the set of all ordered n-tuples satisfying the given system. Suppose we want to find all $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfying the equations:

$$\begin{aligned} 2x + y &= 8 \\ x - 3y &= -3 \end{aligned}$$

We can do this algebraically to get:

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Consistency

Let S be a linear system. We say S is **consistent** if S has at least 1 solution. Otherwise, we say S is **inconsistent**. Any linear system S has three possibilities.

1. 0 solutions
2. 1 solutions
3. ∞ - many solutions

For example:

$$\begin{aligned} x - y &= 1 \\ x - y &= 4 \end{aligned}$$

In this example, $x - y$ cannot be 2 different things. The solution set here is $\emptyset$, thus this system is inconsistent.

Equivalence

Let S and S' be 2 linear systems. If S and S' have the same solutions, they are **equivalent**. As a corollary, every inconsistent system is equivalent.

Row Echelon Form

For the system:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$x_1, x_2, \ldots, x_n$ are variables and a_{ij} are constants. The variables can be represented as:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The coefficients can be represented using the coefficient matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

The augmented matrix $A' = [A|\vec{b}]$ where:

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A matrix A is in row echelon form is:

1. any row of all 0's occurs at the bottom of the matrix.
2. in each nonzero row, the 1st nonzero entry (leading entry) is in a column to the left of any leading entries below it.

We can use row echelon form to solve a linear equation.

Examples

Solve the linear system represented by:

$$A = \left[\begin{array}{cc|c} 2 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} 2x + 4y &= 1 \\ -y &= 2 \\ 2x + 4(-2) &= 1 \\ 2x &= 9 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{9}{2} \\ -2 \end{bmatrix} \end{aligned}$$

Solve the linear system represented by:

$$A = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{array}\right]$$

$$x = 1 \quad y = 5 \quad 0 = 4$$

This system is inconsistent and has no solution.

Example

Consider:

$$\begin{aligned} 2x + y - z &= 3 \\ x + 5z &= 1 \\ -x + 3y - 2z &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 5 \\ -1 & 3 & -2 \end{bmatrix}$$

$$A' = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 1 & 0 & 5 & 1 \\ -1 & 3 & -2 & 0 \end{array}\right]$$

Putting a Matrix into Reduced Echelon Form

We will use **elementary row operations** to do this:

1. We can multiply a row by a nonzero scalar.
2. We can interchange two rows.
3. We can add a multiple of 1 row to another row.

Example

$$A = \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix}$$

$$\begin{aligned} -2R_1 + R_2 &\to R_2 \\ -2R_1 + R_3 &\to R_3 \\ R_1 + R_4 &\to R_4 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$A = \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix}$$

$$3R_2 + R_4 \to R_4$$

$$A = \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 0 & 29 & 29 & -5 \end{bmatrix}$$

$$\frac{1}{8}R_3 \to R_3$$

$$A = \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 29 & 29 & -5 \end{bmatrix}$$

$$-29R_3 + R_4 \to R_4$$

$$A = \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix}$$

This process is called Gaussian elimination. The **rank** of the resulting matrix A (number of nonzero rows) is 4.

Example

Solve the system:

$$\begin{aligned} 2x_2 + 3x_3 &= 8 \\ 2x_1 + 3x_2 + x_3 &= 5 \\ x1 - x_2 - 2x_3 &= -5 \end{aligned}$$

$$A' = \left[\begin{array}{ccc|c} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{array}\right]$$

$$R_1 \leftrightarrow R_3$$

$$A' = \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 2 & 3 & 1 & 5 \\ 0 & 2 & 3 & 8 \end{array}\right]$$

$$-2R_1 + R_2 \to R_2$$

$$A' = \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 2 & 3 & 8 \end{array}\right]$$

$$\frac{1}{5}R_2 \to R_2$$

$$A' = \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 8 \end{array}\right]$$

$$-2R_2 + R_3 \to R_3$$

$$A' = \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{aligned} x - y - 3z &= -5 \\ y + z &= 3 \\ z &= 2 \end{aligned}$$

This resolve to the solution:

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Reduced Row Echelon Form

A matrix is in **reduced row echelon form** if:

1. it is in reduced echelon form.
2. the leading entries of each row equal 1.
3. all other entries in a column with a leading 1 are zero.

Row echelon forms for a linear equation are not unique, while reduced row echelon forms are unique.

Example

Extending off the previous example:

$$A' = \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{aligned} -R_3 + R_2 &\to R_2 \\ 2R_3 + R_1 &\to R_1 \end{aligned}$$

$$A' = \left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_2 + R_1 \to R_1$$

$$A' = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Example

Solve the system:

$$\begin{aligned} w - x - y + 2z &= 1 \\ 2w - 2x - y + 3z &= 3 \\ -w + x - y &= -3 \end{aligned}$$

$$A' = \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right]$$

$$\begin{aligned} -2R_1 + R_2 &\to R_2 \\ R_1 + R_3 &\to R_3 \end{aligned}$$

$$A' = \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right]$$

$$2R_2 + R_3 \to R_3$$

$$A' = \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 + R_1 \to R_1$$

$$A' = \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} w - x + z &= 2 \\ y - z &= 1 \end{aligned}$$

Let $x = s$ and $y = t$:

$$\vec{x} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s - t + 2 \\ s \\ 1 + t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} (s, t \in \mathbb{R})$$

Rank Theorem

Suppose we have a consistent linear system. Suppose we have n variables. If A is the coefficient matrix:

$$\text{the number of free variables} = n - rank(A)$$

Example

Solve the system:

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 3 \\ x_1 + 2x_2 - x_3 &= -3 \\ 2x_2 - 2x_3 &= 1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & -2 & 1 \end{array}\right]$$

$$-R_1 + R_2 \to R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 2 & -2 & 1 \end{array}\right]$$

$$\frac{1}{2}R_2 \to R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & -2 & 1 \end{array}\right]$$

$$-2R_2 + R_3 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

This system is inconsistent and there is no solution.

Homogeneous Linear Systems

In a homogeneous linear system, all constant terms on the right side are 0.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}$$

For example:

$$\begin{aligned} 2x + 3y - z &= 0 \\ -x + 5y + 2z &= 0 \end{aligned}$$

Theorem: In a homogeneous system with n variables, if we have m equations with $m < n$, then the system has infinitely many solutions.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech