

Linear Algebra

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Lines in \mathbb{R}^2 and \mathbb{R}^3

In \mathbb{R}^2 , a line has the following general equation:

$$ax + by = c$$

If $b \neq 0$:

$$\begin{aligned} by &= -ax + c \\ y &= \frac{-a}{b}x + \frac{c}{b} \end{aligned}$$

Consider the line l given by:

$$\begin{aligned} 2x + y &= 0 \\ y &= -2x \end{aligned}$$

This is a line going through the origin with a slope of -2. We can represent the line as:

$$2x + y = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \vec{n} \cdot \vec{x} = 0$$

For any line $ax + by = 0$, we can write it as:

$$\vec{n} \cdot \vec{x} = 0$$

Vector Form of a Line

Let $x = t$:

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ -2t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

In this example, $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is referred to as \vec{d} , the direction vector for l .

Parametric Form of a Line

$$\begin{aligned}x &= t \\y &= -2t\end{aligned}$$

given $t \in \mathbb{R}$.

Normal Form of a Line

Suppose line l is described by $2x + y = 5$ (general form of a line). Let \vec{x} be a general point on l and let \vec{p} be a fixed point on l .

$$\begin{aligned}\vec{n} \cdot (\vec{x} - \vec{p}) &= 0 \\ \vec{n} \cdot \vec{x} - \vec{n} \cdot \vec{p} &= 0 \\ \vec{n} \cdot \vec{x} &= \vec{n} \cdot \vec{p}\end{aligned}$$

In vector form, this is represented as:

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 5 - 2t \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

In parametric form, this is represented as:

$$\begin{aligned}x &= t \\y &= 5 - 2t\end{aligned}$$

given $t \in \mathbb{R}$.

Example

Find the vector and parametric equations for l in \mathbb{R}^3 given l goes through $P = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

and has direction vector $\vec{d} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$.

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$$

$$l = \begin{cases} x &= 1 + 5t \\ y &= 2 - t \\ z &= -1 + 3t \end{cases}$$

Example

It is possible to determine a line by knowing 2 points on it. Find the equation of the line in \mathbb{R}^3 given points $P = (-1, 5, 0)$ and $Q = (2, 1, 1)$.

$$\overrightarrow{PQ} = \langle 3, -4, 1 \rangle$$

$$\vec{x} = \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$

$$l = \begin{cases} x &= -1 + 3t \\ y &= 5 - 4t \\ z &= t \end{cases}$$

Planes in \mathbb{R}^3

The general form of a plane can be written as:

$$ax + by + cz = d$$

We can rewrite this as a dot product:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = d$$

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0$$

$$\therefore \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$$

Vector form of a Plane

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{p} + s\vec{u} + t\vec{v}$$

where s, t are parameters and \vec{p} is a fixed point on the plane.

Example

Find the normal and general forms of plane \mathbb{P} where point $P = (6, 0, 1)$ is on \mathbb{P} with normal vector:

$$\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Normal form:

$$\begin{aligned} \vec{n} \cdot \vec{x} &= \vec{n} \cdot \vec{p} \\ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

General form:

$$\begin{aligned} x + 2y + 3z &= 6 + 0 + 3 \\ x + 2y + 3z &= 9 \end{aligned}$$

And for good measure, the vector form. Let $y = s$ and $z = t$:

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2s - 3t + 9 \\ s \\ t \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech