

Linear Algebra

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Mathematical Induction

We will try to prove that some statement $P(n)$ for all integers $n \geq n_0$.

1. (Basis) Show $P(n_0)$ is true.
2. (Induction) Induction Hypothesis: We assume $P(n)$ is true. We must show $P(n+1)$ is also true.
3. We then conclude $P(n)$ is true for all integers $n \geq n_0$.

Example

Show that:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = P(n)$$

1. (Basis): $n = 1$

$$1 = \frac{1(1+1)}{2}$$

2. (Induction) Assume $P(n)$ is true. Show $P(n+1)$ is also true.

$$\begin{aligned} 1 + 2 + 3 + \cdots + n + (n+1) &= \frac{(n+1)(n+1+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

Show that the left and right hand sides are equal.

$$\begin{aligned}(1 + 2 + 3 + \cdots + n) + (n + 1) &= \frac{n(n + 1)}{2} + \frac{2(n + 1)}{2} \\ &= \frac{n^2 + n + 2n + 2}{2} \\ &= \frac{n^2 + 3n + 2}{2} \\ &= \frac{(n + 1)(n + 2)}{2}\end{aligned}$$

Example

Define:

$$f(n) = \begin{cases} 1 & n = 0 \\ nf(n - 1) & n > 0 \end{cases}$$

$f(n)$ is the factorial function $f(n) = n!$. Prove it for all $n \geq 0$.

1. (Basis): $n = 0$

$$f(0) \stackrel{?}{=} 0!$$

$$f(0) = 1 = 0! \quad \text{by definition}$$

2. (Induction): Assume it is true for n , assume $f(n) = n!$. Show that $f(n + 1) = (n + 1)!$.

$$\begin{aligned}f(n + 1) &= (n + 1)f(n) \\ &= (n + 1)(n!) \\ &= (n + 1)!\end{aligned}$$

$f(n)$ is true for all integers $n \geq 0$.

Exercise 63

Show that:

$$\vec{u} \cdot \vec{v} = \frac{1}{4} \|\vec{u} + \vec{v}\|^2 - \frac{1}{4} \|\vec{u} - \vec{v}\|^2$$

for all $\vec{u}, \vec{v} \in \mathbb{R}^2$.

$$\begin{aligned}\frac{1}{4}\|\vec{u} + \vec{v}\|^2 - \frac{1}{4}\|\vec{u} - \vec{v}\|^2 &= \frac{1}{4}((\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})) - \frac{1}{4}((\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})) \\ &= \frac{1}{4}(\vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v}) - \frac{1}{4}(\vec{u} \cdot \vec{u} - 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v}) \\ &= \frac{2}{4}\vec{u} \cdot \vec{v} + \frac{2}{4}\vec{u} \cdot \vec{v} \\ &= \vec{u} \cdot \vec{v}\end{aligned}$$

Exercise 74

Show that:

$$\left\| \sum_{i=1}^n \vec{v}_i \right\| \leq \sum_{i=1}^n \|\vec{v}_i\|$$

1. (Basis): $n = 2$:

$$\|\vec{v}_1 + \vec{v}_2\| \leq \|\vec{v}_1\| + \|\vec{v}_2\|$$

2. (Induction): Assume $P(n)$ is true. Show that $P(n+1)$ is also true.

$$\begin{aligned}\left\| \sum_{i=1}^n \vec{v}_i \right\| &\leq \sum_{i=1}^{n+1} \|\vec{v}_i\| \\ \left\| \left(\sum_{i=1}^n \vec{v}_i \right) + \vec{v}_{n+1} \right\| &\leq \left\| \sum_{i=1}^n \vec{v}_i \right\| + \|\vec{v}_{n+1}\| \\ &\leq \sum_{i=1}^n \|\vec{v}_i\| + \|\vec{v}_{n+1}\| \\ &= \sum_{i=1}^{n+1} \|\vec{v}_i\|\end{aligned}$$

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech