Differential Equations: Homework 11

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Section 7.5

Exercise 2

Solve the given initial value problem using the method of Laplace transforms.

$$
\begin{aligned}
y'' - y' - 2y &= 0 \quad y(0) = -2 \quad y'(0) = 5 \\
\mathcal{L}\{y'' - y' - 2y\} &= \mathcal{L}\{0\} \\
\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} &= 0 \\
\left[s^2Y(s) - sy(0) - y'(0)\right] - \left[sY(s) - y(0)\right] - 2(Y(s)) &= 0 \\
s^2Y(s) + 2s - 5 - sY(s) - 2 - 2Y(s) &= 0 \\
s^2Y(s) - sY(s) - 2Y(s) &= 7 - 2s \\
Y(s)(s^2 - s - 2) &= 7 - 2s \\
Y(s) &= \frac{7 - 2s}{s^2 - s - 2} \\
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
\frac{A}{s-2} + \frac{B}{s+1} &= \frac{7 - 2s}{(s-2)(s+1)} \\
A(s+1) + B(s-2) &= 7 - 2s \\
Let: s = -1 \quad B &= -3 \\
Let: s = 2 \quad A &= 1 \\
y(t) &= \mathcal{L}^{-1}\{\frac{1}{s-2} - \frac{3}{s+1}\} \\
&= \mathcal{L}^{-1}\{\frac{1}{s-2}\} - 3\mathcal{L}^{-1}\{\frac{1}{s+1}\} \\
&= \mathrm{e}^{2t} - 3\mathrm{e}^{-t}
\end{aligned}
$$

Exercise 3

Solve the given initial value problem using the method of Laplace transforms.

$$\begin{aligned}
y'' + 6y' + 9y &= 0 \quad y(0) = -1 \quad y'(0) = 6 \\
\mathcal{L}\{y'' + 6y' + 9y\} &= \mathcal{L}\{0\} \\
\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} &= 0 \\
\left[s^2Y(s) - sy(0) - y'(0)\right] + 6\left[sY(s) - y(0)\right] + 9(Y(s)) &= 0 \\
s^2Y(s) + s - 6 + 6sY(s) + 6 + 9Y(s) &= 0 \\
s^2Y(s) + 6sY(s) + 9Y(s) &= -s \\
Y(s)(s^2 + 6s + 9) &= -s \\
Y(s) &= \frac{-s}{s^2 + 6s + 9} \\
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
\frac{A}{(s+3)^2} + \frac{B}{s+3} &= \frac{-s}{(s+3)^2} \\
A + B(s+3) &= -s \\
Let : s = -3 \quad A &= 3 \\
Let : s = 1 \quad B &= -1 \\
y(t) &= \mathcal{L}^{-1}\{\frac{3}{(s+3)^2} - \frac{1}{s+3}\} \\
&= 3\mathcal{L}^{-1}\{\frac{1}{(s+3)^2}\} - \mathcal{L}^{-1}\{\frac{1}{s+3}\} \\
&= 3t\mathrm{e}^{-3t} - \mathrm{e}^{-3t}
\end{aligned}$$

Exercise 4

Solve the given initial value problem using the method of Laplace transforms.

$$\begin{aligned}
y'' + 6y' + 5y &= 12\mathrm{e}^t \quad y(0) = -1 \quad y'(0) = 7 \\
\mathcal{L}\{y'' + 6y' + 5y\} &= \mathcal{L}\{12\mathrm{e}^t\} \\
\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} &= 12\mathcal{L}\{\mathrm{e}^t\} \\
\left[s^2Y(s) - sy(0) - y'(0)\right] + 6\left[sY(s) - y(0)\right] + 5Y(s) &= 12\frac{1}{s-1} \\
s^2Y(s) + s - 7 + 6sY(s) + 6 + 5Y(s) &= \frac{12}{s-1} \\
s^2Y(s) + 6sY(s) + 5Y(s) &= \frac{12}{s-1} - s + 1 \\
Y(s)(s^2 + 6s + 5) &= \frac{12}{s-1} - s + 1 \\
Y(s) &= \frac{12}{(s-1)(s+5)(s+1)} - \frac{s-1}{(s+5)(s+1)} \\
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
\frac{A}{s-1} + \frac{B}{s+5} + \frac{C}{s+1} &= \frac{12}{(s-1)(s+5)(s+1)}
\end{aligned}$$

$$A(s+5)(s+1)+B(s-1)(s+1)+C(s-1)(s+5)=12$$

$$\begin{aligned}
Let: s=-5 \quad B &= \frac{1}{2}\\
Let: s=-1 \quad C &= -\frac{3}{2}\\
Let: s=1 \quad A &= 1\\
\frac{D}{s+5}+\frac{E}{s+1} &= \frac{s-1}{(s+5)(s+1)}\\
D(s+1)+E(s+5) &= s-1\\
Let: s=-5 \quad D &= \frac{3}{2}\\
Let: s=-1 \quad E &= -\frac{1}{2}\\
Y(s) &= \frac{1}{s-1}+\frac{1}{2}\frac{1}{s+5}-\frac{3}{2}\frac{1}{s+1}-\frac{3}{2}\frac{1}{s+5}+\frac{1}{2}\frac{1}{s+1}\\
y(t) &= \mathcal{L}^{-1}\{\frac{1}{s-1}-\frac{1}{s+5}-\frac{1}{s+1}\}\\
&= \mathcal{L}^{-1}\{\frac{1}{s-1}\}-\mathcal{L}^{-1}\{\frac{1}{s+5}\}-\mathcal{L}^{-1}\{\frac{1}{s+1}\}\\
&= \mathrm{e}^{t}-\mathrm{e}^{-5t}-\mathrm{e}^{-t}
\end{aligned}$$

Exercise 7

Solve the given initial value problem using the method of Laplace transforms.

$$\begin{aligned}
y''-7y'+10y &= 9\cos(t)+7\sin(t) \quad y(0)=5 \quad y'(0)=-4\\
\mathcal{L}\{y''-7y'+10y\} &= \mathcal{L}\{9\cos(t)+7\sin(t)\}\\
\mathcal{L}\{y''\}-7\mathcal{L}\{y'\}+10\mathcal{L}\{y\} &= 9\mathcal{L}\{\cos(t)\}+7\mathcal{L}\{\sin(t)\}\\
\Big[s^2Y(s)-sy(0)-y'(0)\Big]-7\Big[sY(s)-y(0)\Big]+10Y(s) &= 9\frac{s}{s^2+1}+7\frac{1}{s^2+1}\\
s^2Y(s)-5s+4-7sY(s)+35+10Y(s) &= \frac{9s+7}{s^2+1}\\
s^2Y(s)-7sY(s)+10Y(s) &= \frac{9s+7}{s^2+1}+5s-39\\
Y(s) &= \frac{9s+7}{(s^2+1)(s-2)(s-5)}+\frac{5s-39}{(s-2)(s-5)}\\
&= \frac{5s^3-39s^2+14s-32}{(s^2+1)(s-2)(s-5)}\\
\frac{As+B}{s^2+1}+\frac{C}{s-2}+\frac{D}{s-5} &= \frac{5s^3-39s^2+14s-32}{(s^2+1)(s-2)(s-5)}
\end{aligned}$$

$$(As+B)(s-2)(s-5)+C(s^2+1)(s-5)+D(s^2+1)(s-2)=5s^3-39s^2+14s-32$$

$$\begin{aligned}
&Let : s = 2 \quad C = 8\\
&Let : s = 5 \quad D = -4\\
&Let : s = 0 \quad B = 0\\
&Let : s = 1 \quad A = 1
\end{aligned}$$

$$\begin{aligned}
Y(s) &= \frac{s}{s^2+1} + \frac{8}{s-2} - \frac{4}{s-5}\\
y(t) &= \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} + 8\mathcal{L}^{-1}\{\frac{1}{s-2}\} - 4\mathcal{L}^{-1}\{\frac{1}{s-5}\}\\
&= \cos(t) + 8\mathrm{e}^{2t} - 4\mathrm{e}^{5t}
\end{aligned}$$

Exercise 10

Solve the given initial value problem using the method of Laplace transforms.

$$\begin{aligned}
y'' - 4y &= 4t - 8\mathrm{e}^{-2t} \quad y(0) = 0 \quad y'(0) = 5\\
\mathcal{L}\{y'' - 4y\} &= \mathcal{L}\{4t - 8\mathrm{e}^{-t}\}\\
\mathcal{L}\{y''\} - 4\mathcal{L}\{y\} &= 4\mathcal{L}\{t\} - 8\mathcal{L}\{\mathrm{e}^{-t}\}\\
\Big[s^2Y(s) - sy(0) - y'(0)\Big] - 4Y(s) &= 4\frac{1}{s^2} - 8\frac{1}{s+1}\\
s^2Y(s) - 5 - 4Y(s) &= \frac{4}{s^2} - \frac{8}{s+1}\\
Y(s)(s^2-4) &= \frac{4}{s^2} - \frac{8}{s+2} + 5\\
Y(s) &= \frac{4}{s^2(s^2-4)} - \frac{8}{(s+2)(s^2-4)} + \frac{5}{s^2-4}\\
&= \frac{5s^3+2s^2+4s+8}{s^2(s^2-4)(s+2)}\\
&= \frac{5s^3+2s^2+4s+8}{s^2(s+2)(s-2)(s+2)}\\
\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{(s+2)^2} + \frac{E}{s-2} &= \frac{5s^3+2s^2+4s+8}{s^2(s+2)(s-2)(s+2)}
\end{aligned}$$

$$As(s+2)^2(s-2) + B(s+2)^2(s-2) + Cs^2(s+2)(s-2) + Ds^2(s-2) + Es^2(s+2)^2 = 5s^3 + 2s^2 + 4s + 8$$

$$\begin{aligned}
Let : s = -2 \quad & D = 2\\
Let : s = 2 \quad & E = 1\\
Let : s = 0 \quad & B = -1\\
& A = 0 \quad \text{via system of equations}\\
& C = -1
\end{aligned}$$

$$\begin{aligned}
Y(s) &= -\frac{1}{s^2} - \frac{1}{s+2} + \frac{2}{(s+2)^2} + \frac{1}{s-2}\\
y(t) &= -\mathcal{L}^{-1}\{\frac{1}{s^2}\} - \mathcal{L}^{-1}\{\frac{1}{s+2}\} + 2\mathcal{L}^{-1}\{\frac{1}{(s+2)^2}\} + \mathcal{L}^{-1}\{\frac{1}{s-2}\}\\
&= -t - \mathrm{e}^{-2t} + 2t\mathrm{e}^{-2t} + \mathrm{e}^{2t}
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech