Differential Equations: Homework 4

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Section 2.4

Exercise 1

Classify the equation as separable, linear, exact, or none of these. Notice that some equations may have more than one classification.

$$
\begin{aligned}
(x^2y + x^4\cos(x))\ \mathrm{d}x - x^3\ \mathrm{d}y &= 0 \\
M(x,y) = x^2y + x^4\cos(x) \quad N(x,y) &= -x^3 \\
\frac{\partial M}{\partial y} = x^2 \quad \frac{\partial N}{\partial x} &= -3x^2 \\
\frac{\partial M}{\partial y} &\neq \frac{\partial N}{\partial x}
\end{aligned}
$$

Not an exact equation.

$$
\begin{aligned}
(x^2y + x^4\cos(x))\ \mathrm{d}x - x^3\ \mathrm{d}y &= 0 \\
x^3\ \mathrm{d}y &= (x^2y + x^4\cos(x))\ \mathrm{d}x \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^2y + x^4\cos(x)}{x^3} \\
\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} &= x\cos(x)
\end{aligned}
$$

Linear.

Exercise 3

Classify the equation as separable, linear, exact, or none of these. Notice that some equations may have more than one classification.

$$
\begin{aligned}
\sqrt{-2y - y^2}\ \mathrm{d}x + x\ \mathrm{d}y &= 0 \\
M(x,y) = \sqrt{-2y - y^2} \quad N(x,y) &= x \\
\frac{\partial M}{\partial y} &= (-2 - 2y)\frac{1}{\sqrt{-2y - y^2}} \\
\frac{\partial N}{\partial x} &= 1 \\
\frac{\partial M}{\partial y} &\neq \frac{\partial N}{\partial x}
\end{aligned}
$$

Not an exact equation.

$$
\begin{aligned}
\sqrt{-2y - y^2}\ \mathrm{d}x + x\ \mathrm{d}y &= 0 \\
\sqrt{-2y - y^2}\ \mathrm{d}x &= -x\ \mathrm{d}y \\
-\frac{1}{x}\ \mathrm{d}x &= \frac{1}{\sqrt{-2y - y^2}}\ \mathrm{d}y
\end{aligned}
$$

Separable.

Exercise 5

Classify the equation as separable, linear, exact, or none of these. Notice that some equations may have more than one classification.

$$\begin{aligned} xy\ \mathrm{d}x + \ \mathrm{d}y &= 0 \\ M(x,y) = xy \quad N(x,y) &= 1 \\ \frac{\partial M}{\partial y} = x \quad \frac{\partial N}{\partial x} &= 0 \\ \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \end{aligned}$$

Not an exact equation.

$$\begin{aligned} xy\ \mathrm{d}x + \ \mathrm{d}y &= 0 \\ x\ \mathrm{d}x &= -\frac{1}{y}\ \mathrm{d}y \end{aligned}$$

Separable.

Exercise 7

Classify the equation as separable, linear, exact, or none of these. Notice that some equations may have more than one classification.

$$\begin{aligned} \Big[2x + y\cos(xy)\Big]\ \mathrm{d}x + \Big[x\cos(xy) - 2y\Big]\ \mathrm{d}y &= 0 \\ M(x,y) = 2x + y\cos(xy) \quad N(x,y) &= x\cos(xy) - 2y \\ \frac{\partial M}{\partial y} = \cos(xy) - xy\sin(xy) \quad \frac{\partial N}{\partial x} &= \cos(xy) - xy\sin(xy) \\ \frac{\partial M}{\partial y} &= \frac{\partial N}{\partial x} \end{aligned}$$

This is an exact equation. Not linear or separable.

Exercise 9

Determine whether the equation is exact. If it is, then solve it.

$$\begin{aligned} (2xy + 3)\ \mathrm{d}x + (x^2 - 1)\ \mathrm{d}y &= 0 \\ M(x,y) = 2xy + 3 \quad N(x,y) &= x^2 - 1 \\ \frac{\partial M}{\partial y} = 2x \quad \frac{\partial N}{\partial x} &= 2x \\ \frac{\partial M}{\partial y} &= \frac{\partial N}{\partial x} \end{aligned}$$

This is an exact equation.

$$\begin{aligned} f(x,y) &= \int M(x,y)\ \mathrm{d}x + h(y) \\ &= \int (2xy+3)\ \mathrm{d}x + h(y) \\ &= x^2y + 2x + h(y) \\ \frac{\partial f}{\partial y} &= x^2 + h'(y) = N(x,y) \\ h'(y) &= -1 \\ h(y) &= -y + c \\ f(x,y) &= x^2y + 2x - y - c \end{aligned}$$

Exercise 11

Determine whether the equation is exact. If it is, then solve it.

$$(\mathrm{e}^x \sin(y) - 3x^2)\ \mathrm{d}x + (\mathrm{e}^x \cos(y) + \frac{y^{-2/3}}{3})\ \mathrm{d}y = 0$$

$$M(x,y) = \mathrm{e}^x \sin(y) - 3x^2 \quad N(x,y) = \mathrm{e}^x \cos(y) + \frac{y^{-2/3}}{3}$$

$$\frac{\partial M}{\partial y} = \mathrm{e}^x \cos(y) \quad \frac{\partial N}{\partial y} = \mathrm{e}^x \cos(y)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

This is an exact equation.

$$\begin{aligned} f(x,y) &= \int M(x,y)\ \mathrm{d}x + h(y) \\ &= \int (\mathrm{e}^x \sin(y) - 3x^2)\ \mathrm{d}x + h(y) \\ &= \mathrm{e}^x \sin(y) - x^3 + h(y) \\ \frac{\partial f}{\partial y} &= \mathrm{e}^x \cos(y) + h'(y) = N(x,y) \\ h'(y) &= \frac{y^{-2/3}}{3} \\ h(y) &= -\frac{3}{2}\frac{1}{3}y^{\frac{1}{3}} + c = -\frac{1}{2}y^{\frac{1}{3}} + c \\ f(x,y) &= \mathrm{e}^x \sin(y) - x^3 - \frac{1}{2}y^{\frac{1}{3}} + c \end{aligned}$$

Exercise 13

Determine whether the equation is exact. If it is, then solve it.

$$\mathrm{e}^t(y-t)\ \mathrm{d}t + (1+\mathrm{e}^t)\ \mathrm{d}y = 0$$

$$M(t,y) = \mathrm{e}^t(y-t) \quad N(t,y) = 1 + \mathrm{e}^t$$

$$\frac{\partial M}{\partial y} = \mathrm{e}^t \quad \frac{\partial N}{\partial t} = \mathrm{e}^t$$

This is exactly an equation.

$$\begin{aligned}
f(t,y) &= \int M(x,y)\ \mathrm{d}t + h(y) \\
&= \int (\mathrm{e}^t(y-t))\ \mathrm{d}t + h(y) \\
&= \int (y\mathrm{e}^t - t\mathrm{e}^t)\ \mathrm{d}t + h(y) \\
&= y\mathrm{e}^t - t\mathrm{e}^t + \mathrm{e}^t + h(y) \\
\frac{\partial f}{\partial y} &= \mathrm{e}^t + h'(y) = N(t,y) \\
h'(y) &= 1 \\
h(y) &= y + c \\
f(t,y) &= y\mathrm{e}^t - t\mathrm{e}^t + \mathrm{e}^t + y + c
\end{aligned}$$

Exercise 15

Determine whether the equation is exact. If it is, then solve it.

$$\begin{aligned}
\cos\theta\ \mathrm{d}r - (r\sin\theta - \mathrm{e}^\theta)\ \mathrm{d}\theta = 0 \\
M(r,\theta) = \cos\theta \quad N(r,\theta) = -r\sin\theta + \mathrm{e}^\theta \\
\frac{\partial M}{\partial \theta} = -\sin\theta \quad \frac{\partial N}{\partial r} = -\sin\theta
\end{aligned}$$

This be an exact equation.

$$\begin{aligned}
f(r,\theta) &= \int M(r,\theta)\ \mathrm{d}r + h(\theta) \\
&= \int \cos\theta\ \mathrm{d}r + h(\theta) \\
&= r\cos\theta + h(\theta) \\
\frac{\partial f}{\partial \theta} &= -r\sin\theta + h'(\theta) = N(r,\theta) \\
h'(\theta) &= \mathrm{e}^\theta \\
h(\theta) &= \mathrm{e}^\theta \\
f(r,\theta) &= r\cos\theta + \mathrm{e}^\theta
\end{aligned}$$

Exercise 21

Solve the initial value problem given $y(1) = \pi$.

$$\begin{aligned}
(\frac{1}{x} + 2y^2x)\ \mathrm{d}x + (2yx^2 - \cos(y))\ \mathrm{d}y = 0 \\
M(x,y) = \frac{1}{x} + 2y^2x \quad N(x,y) = 2yx^2 - \cos(y) \\
\frac{\partial M}{\partial y} = 4xy \quad \frac{\partial N}{\partial x} = 4xy
\end{aligned}$$

This is an equation of the exact variety.

$$
\begin{aligned}
f(x,y) &= \int M(x,y)\ \mathrm{d}x + h(y) \\
&= \int (\frac{1}{x} + 2y^2x)\ \mathrm{d}x + h(y) \\
&= \ln(x) + y^2x^2 + h(y) \\
\frac{\partial f}{\partial y} &= 2x^2y + h'(y) = N(x,y) \\
h'(y) &= -\cos(y) \\
h(y) &= -\sin(y) + k \\
f(x,y) &= k = \ln(x) + y^2x^2 - \sin(y) + c \\
&= \ln(1) + \pi^2 1^2 - \sin(\pi) = \pi^2 \\
\pi^2 &= \ln(x) + y^2x^2 - \sin(y)
\end{aligned}
$$

Exercise 23

Solve the initial value problem given $y(0) = -1$.

$$
\begin{aligned}
(\mathrm{e}^t y + t\mathrm{e}^t y)\ \mathrm{d}t + (t\mathrm{e}^t + 2)\ \mathrm{d}y &= 0 \\
\frac{\mathrm{e}^t(1+t)}{t\mathrm{e}^t + 2}\ \mathrm{d}t &= -\frac{1}{y}\ \mathrm{d}y \\
\int \frac{\mathrm{e}^t(1+t)}{t\mathrm{e}^t + 2}\ \mathrm{d}t &= \int -\frac{1}{y}\ \mathrm{d}y \\
\ln(t\mathrm{e}^t + 2) &= -\ln(y) + c \\
t\mathrm{e}^t + 2 &= \frac{1}{y}\mathrm{e}^c \\
y &= \frac{\mathrm{e}^c}{t\mathrm{e}^t + 2} = \frac{k}{t\mathrm{e}^t + 2} \\
-1 &= \frac{k}{0+2} \\
-2 &= k \\
y &= \frac{-2}{t\mathrm{e}^t + 2}
\end{aligned}
$$

Exercise 24

Solve the initial value prolem given $x(1) = 1$.

$$
\begin{aligned}
(\mathrm{e}^t x + 1)\ \mathrm{d}t + (\mathrm{e}^t - 1)\ \mathrm{d}x &= 0 \\
M(t,x) = \mathrm{e}^t x + 1 \quad N(t,x) &= \mathrm{e}^t - 1 \\
\frac{\partial M}{\partial x} = \mathrm{e}^t \quad \frac{\partial N}{\partial t} &= \mathrm{e}^t
\end{aligned}
$$

This is an exact equation.

$$\begin{aligned}
f(t,x) &= \int M(t,x)\ \mathrm{d}t + h(x) \\
&= \int (\mathrm{e}^t x + 1)\ \mathrm{d}t + h(x) \\
&= x\mathrm{e}^t + t + h(x) \\
\frac{\partial f}{\partial x} &= \mathrm{e}^t + h'(x) = N(t,x) \\
h'(x) &= -1 \\
h(x) &= -x + c \\
f(t,x) &= k = x\mathrm{e}^t + t - x \\
k &= 1\mathrm{e}^1 + 1 - 1 = \mathrm{e} \\
\mathrm{e} &= x\mathrm{e}^t + t - x
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech