

Differential Equations: Homework 3

Alvin Lin

January 2018 - May 2018

Section 2.3

Exercise 1

Determine whether the given equation is separable, linear, neither, or both.

$$x^2 \frac{dy}{dx} + \sin(x) - y = 0$$

Not separable and linear.

Exercise 3

Determine whether the given equation is separable, linear, neither, or both.

$$(t^2 + 1) \frac{dy}{dt} = yt - y$$

Separable and non-linear.

Exercise 5

Determine whether the given equation is separable, linear, neither, or both.

$$x \frac{dx}{dt} + t^2 x = \sin(t)$$

Not separable and non-linear.

Exercise 7

Obtain the general solution to the equation:

$$\begin{aligned} \frac{dy}{dx} - y - e^{3x} &= 0 \\ \frac{dy}{dx} - y &= e^{3x} \\ \mu(x) &= e^{\int -1 \, dx} = e^{-x} \\ e^{-x} \frac{dy}{dx} - ye^{-x} &= e^{3x} e^{-x} \\ \int \left(e^{-x} \frac{dy}{dx} - ye^{-x} \right) dx &= \int e^{3x} e^{-x} \, dx \\ ye^{-x} &= \frac{1}{2} e^{2x} + c \end{aligned}$$

Exercise 8

Obtain the general solution to the equation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{x} + 2x + 1 \\ \frac{dy}{dx} - \frac{1}{x}y &= 2x + 1 \\ \mu(x) &= e^{\int \frac{-1}{x} dx} = e^{-\ln(x)} = \frac{1}{x} \\ \frac{1}{x} \frac{dy}{dx} - y &= 2 + \frac{1}{x} \\ \int \left(\frac{1}{x} \frac{dy}{dx} - \frac{1}{x^2}y \right) dx &= \int 2 + \frac{1}{x} dx \\ \frac{y}{x} &= 2x - \ln(x) + c \\ y &= 2x^2 - x \ln(x) + cx\end{aligned}$$

Exercise 9

Obtain the general solution to the equation:

$$\begin{aligned}\frac{dr}{d\theta} + r \tan \theta &= \sec \theta \\ \mu(\theta) &= e^{\int \tan \theta d\theta} = e^{\ln(\sec \theta)} = \sec \theta \\ \sec(\theta) \frac{dr}{d\theta} + r \tan \theta \sec \theta &= \sec^2 \theta \\ \int \left(\sec(\theta) \frac{dr}{d\theta} + r \tan \theta \sec \theta \right) d\theta &= \int \sec^2 \theta d\theta \\ r \sec \theta &= \tan \theta + c \\ r &= \sin \theta + c \cos \theta\end{aligned}$$

Exercise 10

Obtain the general solution to the equation:

$$\begin{aligned}x \frac{dy}{dx} + 2y &= x^{-3} \\ \frac{dy}{dx} + \frac{2}{x}y &= x^{-4} \\ \mu(x) &= e^{\int \frac{2}{x} dx} = e^{2\ln(x)} = x^2 \\ x^2 \frac{dy}{dx} + 2xy &= x^{-2} \\ \int \left(x^2 \frac{dy}{dx} + 2xy \right) dx &= \int x^{-2} dx \\ yx^2 &= -\frac{1}{x} + c \\ y &= -\frac{1}{x^3} + \frac{c}{x^2}\end{aligned}$$

Exercise 11

Obtain the general solution to the equation:

$$\begin{aligned}
 (t + y + 1) dt - dy &= 0 \\
 \frac{dy}{dt} &= t + y + 1 \\
 \frac{dy}{dt} - y &= t + 1 \\
 \mu(t) &= e^{\int -1 dt} = e^{-t} \\
 e^{-t} \frac{dy}{dt} - ye^{-t} &= te^{-t} + e^{-t} \\
 \int \left(e^{-t} \frac{dy}{dt} - ye^{-t} \right) &= \int te^{-t} + e^{-t} dt \\
 ye^{-t} &= -e^{-t}(t + 1) - e^{-t} + c \\
 y &= -t - 1 - 1 + ce^t \\
 &= -t - 2 + ce^t
 \end{aligned}$$

Exercise 15

Obtain the general solution to the equation:

$$\begin{aligned}
 (x^2 + 1) \frac{dy}{dx} + xy - x &= 0 \\
 \frac{dy}{dx} + \frac{x}{x^2 + 1} y &= \frac{x}{x^2 + 1} \\
 \mu(x) &= e^{\int \frac{x}{x^2+1} dx} = e^{\frac{1}{2} \ln(x^2+1)} = \sqrt{x^2 + 1} \\
 \sqrt{x^2 + 1} \frac{dy}{dx} + \frac{x}{\sqrt{x^2 + 1}} y &= \frac{x}{\sqrt{x^2 + 1}} \\
 \int \left(\sqrt{x^2 + 1} \frac{dy}{dx} + \frac{x}{\sqrt{x^2 + 1}} y \right) dx &= \int \frac{x}{\sqrt{x^2 + 1}} dx \\
 y\sqrt{x^2 + 1} &= \sqrt{x^2 + 1} + c \\
 y &= 1 + \frac{c}{\sqrt{x^2 + 1}}
 \end{aligned}$$

Exercise 17

Solve the initial value problem.

$$\begin{aligned}
 \frac{dy}{dx} - \frac{y}{x} &= xe^x \quad y(1) = e - 1 \\
 \mu(x) &= e^{\int -\frac{1}{x} dx} = e^{-\ln(x)} = \frac{1}{x} \\
 \frac{1}{x} \frac{dy}{dx} - \frac{1}{x^2} y &= e^x \\
 \int \left(\frac{1}{x} \frac{dy}{dx} - \frac{1}{x^2} y \right) dx &= \int e^x dx \\
 \frac{y}{x} &= e^x + c \\
 y &= xe^x + cx
 \end{aligned}$$

$$e - 1 = 1e^1 + c1$$

$$c = -1$$

$$y = xe^x - x$$

Exercise 18

Solve the initial value problem.

$$\frac{dy}{dx} + 4y - e^{-x} = 0 \quad y(0) = \frac{4}{3}$$

$$\frac{dy}{dx} + 4y = e^{-x}$$

$$\mu(x) = e^{\int 4 dx} = e^{4x}$$

$$e^{4x} \frac{dy}{dx} + 4ye^{4x} = e^{3x}$$

$$\int \left(e^{4x} \frac{dy}{dx} + 4ye^{4x} \right) dx = \int e^{3x} dx$$

$$ye^{4x} = \frac{1}{3}e^{3x} + c$$

$$y = \frac{1}{3e^x} + \frac{c}{e^{4x}}$$

$$\frac{4}{3} = \frac{1}{3e^0} + \frac{c}{e^0}$$

$$c = 1$$

$$y = \frac{1}{3e^x} + \frac{1}{e^{4x}}$$

Exercise 20

Solve the initial value problem.

$$\frac{dy}{dx} + \frac{3y}{x} + 2 = 3x \quad y(1) = 1$$

$$\frac{dy}{dx} + \frac{3y}{x} = 3x - 2$$

$$\mu(x) = e^{\int \frac{3}{x} dx} = e^{3\ln(x)} = x^3$$

$$x^3 \frac{dy}{dx} + 3x^2y = 3x^4 - 2x^3$$

$$\int \left(x^3 \frac{dy}{dx} + 3x^2y \right) dx = \int 3x^4 - 2x^3 dx$$

$$yx^3 = \frac{3}{5}x^5 - \frac{2}{4}x^4 + c$$

$$y = \frac{3}{5}x^2 - \frac{1}{2}x^2 + \frac{c}{x^3}$$

$$1 = \frac{3}{5} - \frac{1}{2} + c \quad c = \frac{9}{10}$$

$$y = \frac{3}{5}x^2 - \frac{1}{2}x^2 + \frac{9}{10x^3}$$

Section 2.6

Exercise 21

Use the method discussed under “Bernoulli Equations” to solve:

$$\begin{aligned}\frac{dy}{dx} + \frac{y}{x} &= x^2 y^2 \\ v &= y^{-1} \\ \frac{dv}{dx} &= -y^{-2} \frac{dy}{dx} \\ \frac{1}{y^2} \frac{dy}{dx} + \frac{1}{y^2} \frac{y}{x} &= \frac{1}{y^2} x^2 y^2 \\ \frac{1}{y^2} \frac{dy}{dx} + \frac{1}{xy} &= x^2 \\ -\frac{dv}{dx} + \frac{v}{x} &= x^2 \\ \frac{dv}{dx} - \frac{v}{x} &= -x^2 \\ \mu(x) &= e^{\int -\frac{1}{x} dx} = e^{-\ln(x)} = \frac{1}{x} \\ \frac{1}{x} \frac{dv}{dx} - \frac{v}{x^2} &= -x^2 \\ \int \left(\frac{1}{x} \frac{dv}{dx} - \frac{v}{x^2} \right) dx &= \int -x^2 dx \\ \frac{v}{x} &= -\frac{x^3}{3} + c \\ \frac{1}{xy} &= -\frac{x^3}{3} + c\end{aligned}$$

Exercise 22

Use the method discussed under “Bernoulli Equations” to solve:

$$\begin{aligned}\frac{dy}{dx} - y &= e^{2x} y^3 \\ P(x) = -1 \quad Q(x) = e^{2x} \quad n = 3 \quad v = y^{-2} \quad \frac{dv}{dx} &= -\frac{1}{2} y^{-3} \frac{dy}{dx} \\ \frac{dv}{dx} + (1 - n)P(x)v &= (1 - n)Q(x) \\ \frac{dv}{dx} + 2v &= -2e^{2x} \\ \mu(x) &= e^{\int 2 dx} = e^{2x} \\ e^{2x} \frac{dv}{dx} + 2ve^{2x} &= -2e^{4x} \\ \int \left(e^{2x} \frac{dv}{dx} + 2ve^{2x} \right) dx &= \int -2e^{4x} dx \\ ve^{2x} &= \frac{-2}{4} e^{4x} + c \\ \frac{e^{2x}}{y^2} &= \frac{-e^{4x}}{2} + c\end{aligned}$$

Exercise 23

Use the method discussed under “Bernoulli Equations” to solve:

$$\begin{aligned} \frac{dy}{dx} &= \frac{2y}{x} - x^2 y^2 \\ \frac{dy}{dx} - \frac{2y}{x} &= -x^2 y^2 \\ v = y^{-1} \quad \frac{dv}{dx} &= -\frac{1}{y^2} \frac{dy}{dx} \\ \frac{1}{y^2} \frac{dy}{dx} - \frac{1}{y^2} \frac{2y}{x} &= -\frac{1}{y^2} x^2 y^2 \\ \frac{1}{y^2} \frac{dy}{dx} - \frac{2}{xy} &= -x^2 \\ -\frac{dv}{dx} - \frac{2v}{x} &= -x^2 \\ \frac{dv}{dx} + \frac{2v}{x} &= x^2 \\ \mu(x) &= e^{\int \frac{2}{x} dx} = e^{2 \ln(x)} = x^2 \\ x^2 \frac{dv}{dx} + 2vx &= x^4 \\ \int \left(x^2 \frac{dv}{dx} + 2vx \right) dx &= \int x^4 dx \\ vx^2 &= \frac{x^5}{5} + c \\ \frac{x^2}{y} &= \frac{x^5}{5} + c \end{aligned}$$

Exercise 24

Use the method discussed under “Bernoulli Equations” to solve:

$$\begin{aligned} \frac{dy}{dx} + \frac{y}{x-2} &= 5(x-2)y^{\frac{1}{2}} \\ v = y^{\frac{1}{2}} \quad \frac{dv}{dx} &= -\frac{1}{\sqrt{y}} \frac{dy}{dx} \\ \frac{1}{\sqrt{y}} \frac{dy}{dx} + \frac{1}{\sqrt{y}} \frac{y}{x-2} &= \frac{1}{\sqrt{y}} 5(x-2)\sqrt{y} \\ \frac{1}{\sqrt{y}} \frac{dy}{dx} + \frac{\sqrt{y}}{x-2} &= 5x - 10 \\ -\frac{dv}{dx} + \frac{v}{x-2} &= 5x - 10 \\ \frac{dv}{dx} - \frac{v}{x-2} &= 10 - 5x \\ \mu(x) &= e^{\int \frac{-1}{x-2} dx} = e^{-\ln(x-2)} = \frac{1}{x-2} \\ (x-2) \frac{dv}{dx} - \frac{v}{(x-2)^2} &= \frac{10-5x}{x-2} \end{aligned}$$

$$\int \left((x-2) \frac{dv}{dx} - \frac{v}{(x-2)^2} \right) dx = \int \frac{10-5x}{x-2} dx$$

$$v(x-2) = -5x + c$$

$$(x-2)\sqrt{y} = -5x + c$$

Exercise 25

Use the method discussed under “Bernoulli Equations” to solve:

$$\frac{dx}{dt} + tx^3 + \frac{x}{t} = 0$$

$$\frac{dx}{dt} + \frac{x}{t} = -x^3$$

$$v = x^{-2} \quad \frac{dv}{dt} = -\frac{1}{2}x^{-3}$$

$$\frac{1}{x^3} \frac{dx}{dt} + \frac{1}{x^3} \frac{x}{t} = -\frac{1}{x^3} x^3$$

$$\frac{1}{x^3} \frac{dx}{dt} + \frac{1}{x^2 t} = -1$$

$$-\frac{dv}{dt} + \frac{v}{t} = -1$$

$$\frac{dv}{dt} - \frac{v}{t} = 1$$

$$\mu(t) = e^{\int -\frac{1}{t} dt} = e^{-\ln(t)} = \frac{1}{t}$$

$$\frac{1}{t} \frac{dv}{dt} - \frac{v}{t^2} = \frac{1}{t}$$

$$\int \left(\frac{1}{t} \frac{dv}{dt} - \frac{v}{t^2} \right) dt = \int \frac{1}{t} dt$$

$$\frac{v}{t} = \ln(t) + c$$

$$\frac{1}{tx^2} = \ln(t) + c$$

Exercise 26

Use the method discussed under “Bernoulli Equations” to solve:

$$\frac{dy}{dx} + y = e^x y^{-2}$$

$$v = y^3 \quad \frac{dv}{dx} = 3y^2 \frac{dy}{dx}$$

$$y^2 \frac{dy}{dx} + y(y^2) = e^x y^{-2}(y^2)$$

$$\frac{1}{3} \frac{dv}{dx} + v = e^x$$

$$\frac{dv}{dx} + 3v = 3e^x$$

$$\mu(x) = e^{\int 3 dx} = e^{3x}$$

$$e^{3x} \frac{dv}{dx} + 3ve^{3x} = 3e^{4x}$$

$$\int \left(e^{3x} \frac{dv}{dx} + 3ve^{3x} \right) dx = \int 3e^{4x} dx$$

$$ve^{3x} = \frac{3}{4}e^{4x} + c$$

$$y^3 e^{3x} = \frac{3}{4}e^{4x} + c$$

Exercise 27

Use the method discussed under “Bernoulli Equations” to solve:

$$\frac{dr}{d\theta} = \frac{r^2 + 2r\theta}{\theta^2}$$

$$= \frac{r^2}{\theta^2} + \frac{2r}{\theta}$$

$$\frac{dr}{d\theta} - \frac{2r}{\theta} = \frac{r^2}{\theta^2}$$

$$v = r^{-1} \quad \frac{dv}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta}$$

$$\frac{1}{r^2} \frac{dr}{d\theta} - \frac{1}{r^2} \frac{2r}{\theta} = \frac{1}{r^2} \frac{r^2}{\theta^2}$$

$$-\frac{dv}{d\theta} - \frac{2v}{\theta} = \frac{1}{\theta^2}$$

$$\frac{dv}{d\theta} + \frac{2v}{\theta} = -\frac{1}{\theta^2}$$

$$\mu(\theta) = e^{\int \frac{2}{\theta} d\theta} = e^{2\ln(\theta)} = \theta^2$$

$$\theta^2 \frac{dv}{d\theta} + 2v\theta = -1$$

$$\int \left(\theta^2 \frac{dv}{d\theta} + 2v\theta \right) d\theta = \int -1 d\theta$$

$$v\theta^2 = -\theta + c$$

$$\frac{\theta^2}{r} = -\theta + c$$

Exercise 28

Use the method discussed under “Bernoulli Equations” to solve:

$$\frac{dy}{dx} + y^3x + y = 0$$

$$\frac{dy}{dx} + y = -xy^3$$

$$v = y^{-2} \quad \frac{dv}{dx} = -\frac{1}{2}y^{-3} \frac{dy}{dx}$$

$$\frac{1}{y^3} \frac{dy}{dx} + \frac{1}{y^3}y = \frac{1}{y^3} - xy^3$$

$$-2\frac{dv}{dx} + v = -x$$

$$\begin{aligned} \frac{dv}{dx} - \frac{v}{2} &= \frac{x}{2} \\ \mu(x) &= e^{\int -\frac{1}{2} dx} = e^{-\frac{x}{2}} \\ e^{-\frac{x}{2}} \frac{dv}{dx} - e^{-\frac{x}{2}} \frac{v}{2} &= \frac{xe^{\frac{x}{2}}}{2} \\ \int \left(e^{-\frac{x}{2}} \frac{dv}{dx} - \frac{ve^{-\frac{x}{2}}}{2} \right) dx &= \int \frac{xe^{-\frac{x}{2}}}{2} dx \\ ve^{-\frac{x}{2}} &= -e^{-\frac{x}{2}}(x-2) + c \\ \frac{1}{y^2} &= 2-x+c \end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech