Differential Equations

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Homogeneous Linear Equations

Consider:

$$ay'' + by' + cy = f(t)$$

where a, b, c are constants and $a \neq 0$. If $f(t) = 0$:

$$ay'' + by' + cy = 0$$

And this is a second order linear homogeneous equation. Consider $y(t) = \mathrm{e}^{rt}$ is a solution:

$$y'(t) = r\mathrm{e}^{rt}$$
$$y''(t) = r^2\mathrm{e}^{rt}$$

If we substitute:

$$ar^2\mathrm{e}^{rt} + br\mathrm{e}^{rt} + c\mathrm{e}^{rt} = 0$$
$$\mathrm{e}^{rt}\left[ar^2 + br + c\right] = 0$$

To satisfy this equation, we have $ar^2 + br + c = 0$. This is called the **characteristic equation** or the **auxiliary equation**. We can determine the roots of $ar^2 + bc + r = 0$ using the quadratic formula.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Using this equation, there are three cases:

1. When $b^2 - 4ac > 0$, the roots are real and distinct.
2. When $b^2 - 4ac = 0$, the root $r = \frac{-b}{2a}$ is a repeated root.
3. When $b^2 - 4ac < 0$, the roots are complex conjugates $r = \alpha \pm i\beta$ where $i = \sqrt{-1}$.

Case 1: Real and Distinct Roots

We can use the roots to solve the differential equation.

$$\begin{aligned} y'' + y' - 2y &= 0 \\ r^2 + r - 2 &= 0 \\ (r+2)(r-2) &= 2 \\ r_1 = -2 \quad r_1 &= 1 \end{aligned}$$

Using this characteristic equation, we can determine $y_1 = \mathrm{e}^{-2t}$ and $y_2 = \mathrm{e}^{t}$. We can verify y_1 and y_2 are both solutions.

$$\begin{aligned} y_1 &= \mathrm{e}^{-2t} & y_2 &= \mathrm{e}^{t} \\ y_1' &= -2\mathrm{e}^{-2t} & y_2' &= \mathrm{e}^{t} \\ y_1'' &= 4\mathrm{e}^{-2t} & y_2'' &= \mathrm{e}^{t} \end{aligned}$$

$$\begin{aligned} y'' + y' - 2y &= 0 \\ 4\mathrm{e}^{-2t} - 2\mathrm{e}^{-2t} - 2\mathrm{e}^{-2t} &= 0 \\ \mathrm{e}^{t} + \mathrm{e}^{t} - 2\mathrm{e}^{t} &= 0 \end{aligned}$$

Furthermore, $y = y_1 + y_2$ also satisfies the equation.

$$\begin{aligned} y &= \mathrm{e}^{-2t} + \mathrm{e}^{t} \\ y' &= -2\mathrm{e}^{-2t} + \mathrm{e}^{t} \\ y'' &= 4\mathrm{e}^{-2t} + \mathrm{e}^{t} \\ 4\mathrm{e}^{-2t} + \mathrm{e}^{t} - 2\mathrm{e}^{-2t} - 2(\mathrm{e}^{-2t} + \mathrm{e}^{t}) &= 0 \end{aligned}$$

Also, note that $y = c_1\mathrm{e}^{r_1 t} + c_2\mathrm{e}^{r_2 t}$ also satisfies

$$ay'' + by' + cy = 0$$

This is a general solution since it contains two arbitrary constants. If we are given initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_1$, then we can solve for c_1 and c_2 and treat it as an initial value problem. For example:

$$\begin{aligned} y'' + y' &= 0 \\ y(0) = 2 \quad y'(0) &= 1 \end{aligned}$$

The characteristic equation is $r^2 + r = 0$, giving us roots $r_1 = 0$ and $r_2 = -1$. Thus we have the general solution

$$y = c_1 e^{0t} + c_2 e^{-t} = c_1 + c_2 e^{-t}$$

We can use our initial conditions to solve for an explicit solution:

$$\begin{aligned} y &= c_1 + c_2 e^{-t} \\ y' &= -c_2 e^{-t} \\ y(0) &= 2 = c_1 + c_2 \\ y'(0) &= 1 = -c_2 \\ c_2 &= -1 \quad c_1 = 3 \\ y &= 3 - e^{-t} \end{aligned}$$

Example

Solve the initial value problem given $y(0) = 0$ and $y'(0) = 1$:

$$\begin{aligned} y'' + 2y' - y &= 0 \\ r^2 + 2r - 1 &= 0 \end{aligned}$$

$$\begin{aligned} r &= \frac{-2 \pm \sqrt{4 - 4(-1)}}{2} \\ &= -1 \pm \sqrt{2} \\ y &= c_1 e^{(-1+\sqrt{2})t} + c_2 e^{(-1-\sqrt{2})t} \\ y' &= (-1+\sqrt{2}) c_1 e^{(-1+\sqrt{2})t} + (-1-\sqrt{2}) c_2 e^{(-1-\sqrt{2})t} \\ y(0) &= 0 = c_1 + c_2 \\ y'(0) &= 1 = (-1+\sqrt{2}) c_1 + (-2-\sqrt{2}) c_2 \\ c_1 &= \frac{1}{2\sqrt{2}} \quad c_2 = -\frac{1}{2\sqrt{2}} \end{aligned}$$

Existence and Uniqueness

If we have existence and uniqueness for the solution to the initial value problem:

$$\begin{aligned} ay'' + by' + cy &= 0 \\ y(x_0) &= y_0 \\ y'(x_0) &= y_1 \end{aligned}$$

then the solution is valid of $(-\infty, \infty)$. If $y_1(t)$ and $y_2(t)$ are any two solutions that are linearly independent over $(-\infty, \infty)$, then $y = c_1 y_1 + c_2 y_2$ form a fundamental set of solutions. If two functions y_1 and y_2 exist such that one is not a constant multiple of the other, then y_1 and y_2 are linearly independent. For example, $y_1 = \cos(x)$ and $y_2 = \sin(x)$ are linearly independent. $y_1 = \mathrm{e}^{2t}$ and $y_2 = \mathrm{e}^{t}$ are also linearly independent. Consider the Wronskian, W:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

If $W = 0$, then y_1 and y_2 are linearly dependent. Otherwise, y_1 and y_2 are linearly independent.

Example

$$\begin{aligned} y_1 &= \cos(x) \\ y_2 &= \sin(x) \\ W(\cos(x), \sin(x)) &= \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} \\ &= \cos^2(x) - (-\sin^2(x)) = 1 \neq 0 \end{aligned}$$

Example

$$\begin{aligned} y_1 &= 3\mathrm{e}^{2t} \\ y_2 &= \mathrm{e}^{2t} \\ W(y_1, y_2) &= \begin{vmatrix} 3\mathrm{e}^{2t} & \mathrm{e}^{2t} \\ 6\mathrm{e}^{2t} & 2\mathrm{e}^{2t} \end{vmatrix} \\ &= 6\mathrm{e}^{2t} - 6\mathrm{e}^{2t} = 0 \end{aligned}$$

Case 2: Real and Repeated Roots

If the roots of the auxiliary equation are real and repeated ($r = \frac{-b}{2a}$), then $y = c_1 \mathrm{e}^{rt} + c_2 \mathrm{e}^{rt}$. Note that $y_1 = c_1 \mathrm{e}^{rt}$ and $y_2 = c_2 \mathrm{e}^{rt}$ will be linearly dependent. We want another linearly independent solution to

$$ay'' + by' + cy = 0$$

If $y_1 = c_1 e^{rt}$, then $y_2 = c_2 t e^{rt}$ gives a second solution when y_1 and y_2 are linearly independent. We append a factor of t to $y_1 = c_1 e^{rt}$. For this case, our general solution is of the form

$$y = c_1 e^{rt} + c_2 t e^{rt}$$

Example

Solve the initial value problem given $y(0) = 1$ and $y'(0) = 3$:

$$\begin{aligned}
y'' + 10y' + 25 &= 0 \\
r^2 + 10r + 25 &= 0 \\
(r+5) &= 0 \\
r &= -5 \\
y &= c_1 e^{-5t} + c_2 t e^{-5t} \\
y(0) &= 1 = c_1 \\
y' &= -5c_1 e^{-5t} + c_2 e^{-5t} - 5c_2 t e^{-5t} \\
y'(0) &= 3 = -5c_1 + c_2 \\
c_2 &= 8 \\
y &= e^{-5t} + 8t e^{-5t}
\end{aligned}$$

Case 3: Complex Roots

If the roots of the auxiliary equation are complex terms $r = \alpha \pm i\beta$ where $i = \sqrt{-1}$, then as in previous cases:

$$y = e^{(\alpha + i\beta)t}$$

We can start by using the rules of exponents to split the complex term in the exponent.

$$y = e^{\alpha t} e^{i\beta t}$$

We can use Euler's formula to resolve this:

$$e^{it} = \cos\theta + i\sin\theta$$

If we let $\theta = \beta t$, then we have:

$$y = e^{\alpha t}\Big[\cos(\beta t) + i\sin(\beta t)\Big]$$

We can see from this point that the solution is going to oscillate because cosine is a sinusoidal wave. From this, one way to write the general form of the solution is:

$$y = \mathrm{e}^{\alpha t}\Big[c_1\cos(\beta t) + ic_2\sin(\beta t)\Big]$$

Given an initial value problem, this would solve it, but going through it would involve complex arithmetic. Typically, the constants c_1 and c_2 are complex values themselves. However, we can avoid complex arithmetic in this particular situation. The equation we are trying to solve is:

$$ay'' + by' + cy = 0$$

Consider a complex valued function $z = u + iv$ that is a solution to 1.

$$\begin{aligned} z &= u + iv \\ z' &= u' + iv' \\ z'' &= u'' + iv'' \end{aligned}$$

If we substitute this into the function:

$$\begin{aligned} a\Big[u'' + iv''\Big] + b\Big[u' + iv'\Big] + c\Big[u + iv\Big] &= 0 \\ \Big[au'' + bu' + cu\Big] + i\Big[av'' + bv' + cv\Big] &= 0 \end{aligned}$$

If $s + it = 0$, then $s = t = 0$. Therefore, both the real and imaginary parts of the equation above are zero.

$$\begin{aligned} au'' + bu' + cu &= 0 \\ av'' + bv' + cv &= 0 \end{aligned}$$

Since we have manipulated u and v into the forms of our original equation, we can conclude that both u and v are solutions to our differential equation. We can write a general solution as:

$$y(t) = \mathrm{e}^{\alpha t}\Big[c_1\cos(\beta t) + c_2\sin(\beta t)\Big]$$

where $r = \alpha \pm i\beta$ are complex valued roots.

Example

Consider this example:

$$
\begin{aligned}
y'' + 2y' + 2y &= 0 \\
r^2 + 2r + 2 &= 0 \\
r &= \frac{-2 \pm \sqrt{4 - (4)(2)}}{2} \\
r &= -1 \pm i \\
\alpha &= -1 \quad \beta = 1 \\
y(t) &= \mathrm{e}^{-t}\Bigg[c_1 \cos(t) + c_2 \sin(t)\Bigg]
\end{aligned}
$$

If we have the initial values $y(0) = 1$ and $y'(0) = 1$:

$$
\begin{aligned}
y(t) &= \mathrm{e}^{-t}\Bigg[c_1 \cos(t) + c_2 \sin(t)\Bigg] \\
y'(t) &= \mathrm{e}^{-t}\Bigg[-c_1 \sin(t) + c_2 \cos(t)\Bigg] - e^{-t}\Bigg[c_1 \cos(t) + c_2 \sin(t)\Bigg] \\
y(0) &= 1 = \mathrm{e}^{0}\Bigg[c_1 \cos(0) + c_2 \sin(0)\Bigg] \\
&= c_1 \\
y'(0) &= 1 = \mathrm{e}^{0}\Bigg[-c_1 \sin(0) + c_2 \cos(0)\Bigg] - \mathrm{e}^{0}\Bigg[c_1 \cos(0) + c_2 \sin(0)\Bigg] \\
&= c_2 - 1(c_1) \\
c_1 &= 1 \quad c_2 = 2 \\
y &= \mathrm{e}^{-t}\Bigg[\cos(t) + 2\sin(t)\Bigg]
\end{aligned}
$$

The Method of Undetermined Coefficients

Consider:

$$ay'' + by' + cy = f(t) \neq 0$$

This is a second order non-homogeneous equation with constant coefficients. The method of undetermined coefficients applies to functions that are exponentials, polynomials, sines and consines, or products of these types of functions. For example:

$$y'' - 3y' + 2y = 5\mathrm{e}^{3t}$$

We seek a particular solution, y_p. We can try $y_p = A\mathrm{e}^{3t}$ where A is a constant.

$$\begin{aligned}
y_p &= A\mathrm{e}^{3t} \\
y_p' &= 3A\mathrm{e}^{3t} \\
y_p'' &= 9A\mathrm{e}^{3t} \\
9A\mathrm{e}^{3t} - 3(3A\mathrm{e}^{3t}) + 2A\mathrm{e}^{3t} &= 5\mathrm{e}^{3t} \\
9A\mathrm{e}^{3t} - 9A\mathrm{e}^{3t} + 2A\mathrm{e}^{3t} & \\
2A &= 5\mathrm{e}^{3t} \\
2A &= 5 \\
y_p &= \frac{5}{2}\mathrm{e}^{3t}
\end{aligned}$$

Example

Consider another example:

$$\begin{aligned}
y'' - 3y' + 2y &= 5\mathrm{e}^{2t} \\
y_p &= A\mathrm{e}^{2t} \\
y_p' &= 2A\mathrm{e}^{2t} \\
y_p'' &= 4A\mathrm{e}^{2t} \\
4A\mathrm{e}^{2t} - 3(2A\mathrm{e}^{2t}) + 2A\mathrm{e}^{2t} &= 0 \neq 5\mathrm{e}^{2t}
\end{aligned}$$

The key is to consider the associated auxiliary equation.

$$\begin{aligned}
r^2 - 3r + 2 &= 0 \\
(r-2)(r-1) &= 0 \\
r_1 &= 1 \quad r_2 = 2
\end{aligned}$$

In this case, $r = 2$ matched one of the roots of $r^2 - 3r + 2 = 0$. This means that $y = c\mathrm{e}^{2t}$ is a solution to the associated homogeneous equation, and not the given non-homogeneous equation. We append a factor of t:

$$y_p = At\mathrm{e}^{2t}$$

If we try this form:

$$\begin{aligned}
y_p &= At\mathrm{e}^{2t} \\
y_p' &= A\mathrm{e}^{2t} + 2At\mathrm{e}^{2t} \\
y_p'' &= 2A\mathrm{e}^{2t} + 2A\mathrm{e}^{2t} + 4At\mathrm{e}^{2t} \\
&= 4A\mathrm{e}^{2t} + 4At\mathrm{e}^{2t}
\end{aligned}$$

$$\begin{aligned}
(4A\mathrm{e}^{2t} + 4At\mathrm{e}^{2t}) - 3(A\mathrm{e}^{2t} + 2At\mathrm{e}^{2t}) + 2At\mathrm{e}^{2t} &= 5\mathrm{e}^{2t} \\
4A\mathrm{e}^{2t} + 4At\mathrm{e}^{2t} - A\mathrm{e}^{2t} - 6At\mathrm{e}^{2t} + 2At\mathrm{e}^{2t} &= 5\mathrm{e}^{2t} \\
A\mathrm{e}^{2t} &= 5\mathrm{e}^{2t} \\
A &= 5 \\
y_p &= 5\mathrm{e}^{2t}
\end{aligned}$$

Example

$$\begin{aligned}
y'' + 3y' + 2y &= 3\mathrm{e}^{0t} \\
r^2 + 3r + 2 &= 0 \\
(r+1)(r+2) &= 0 \\
r_1 = -1 \quad r_2 &= -2
\end{aligned}$$

Neither of these solutions match $r = 0$, so we do not need to append a term t. Consider y_p to be a polynomial of degree 1.

$$\begin{aligned}
y_p &= At + B \\
y_p' &= A \\
y_p'' &= 0
\end{aligned}$$

Substituting these into our differential equation allows us to solve for an explicit solution:

$$
\begin{aligned}
0 + 3A + 2(At + B) &= 3t \\
(3A)t + (3A + 2B) &= 3t \\
2A &= 3 \quad 3A + 2B = 0 \\
B &= -\frac{9}{4} \\
y_p &= \frac{3}{2}t - \frac{9}{4}
\end{aligned}
$$

Example

$$
\begin{aligned}
y'' - y &= 10\mathrm{e}^{-t} \\
r &= -1 \\
r^2 - 1 &= 0 \\
r_1 &= 1 \quad r_2 = -1
\end{aligned}
$$

Since we have a matching root, we append a factor of t to the solution:

$$
\begin{aligned}
y_p &= At\mathrm{e}^{-t} \\
y_p' &= A\mathrm{e}{-t} - At\mathrm{e}^{-t} \\
y_p'' &= -A\mathrm{e}^{-t} - (A\mathrm{e}^{-t} - At\mathrm{e}^{-t}) \\
&= -2A\mathrm{e}^{-t} + At\mathrm{e}^{-t} \\
(-2A\mathrm{e}^{-t} + At\mathrm{e}^{-t}) - (At\mathrm{e}^{-t}) &= 10\mathrm{e}^{-t} \\
-2A\mathrm{e}^{-t} &= 10\mathrm{e}^{-t} \\
-2A &= 10 \\
A &= 5 \\
y_p &= -5t\mathrm{e}^{-t}
\end{aligned}
$$

Example

$$\begin{aligned} y'' + 17y &= -7\mathrm{e}^{0t} \\ r &= 0 \\ r^2 + 17 &= 0 \\ r_1 &= 0 + 17i \quad r_2 = 0 - 17i \\ y_p &= A \\ y_p' &= y_p'' = 0 \\ 0 + 17A &= -7 \\ A &= -\frac{7}{17} \\ y_p &= -\frac{7}{17} \end{aligned}$$

Methodology

The method of undetermined coefficients is a method to find a particular solution y_p to $ay'' + by' + cy = f(t) \neq 0$.

1. When $f(t) = Ct^m\mathrm{e}^{rt}$ where C is any constant, r is a real number, and m is a non-negative integer, the general form of y_p will be:

$$y_p(t) = t^s \left[A_m t^m + A_{m-1}t^{m-1} + \cdots + A_1 t + A_0 \right] \mathrm{e}^{rt}$$

   where $s = 0$ if r doesn't match the roots of the characteristic equation, $s = 1$ if r is a match to the roots of the characteristic equation and the roots are non-repeated, and $s = 2$ if r is a repeated real root of the corresponding auxiliary equation.

2. When $f(t) = Ct^m\mathrm{e}^{\alpha t}\cos(\beta t)$ or $f(t) = Ct^m\mathrm{e}^{\alpha t}\sin(\beta t)$ where m is a positive integer, C is any constant, α is real, and β is a nonzero real number, the form of y_p will be:

$$\begin{aligned} y_p(t) = t^s &\left[A_m t^m + A_{m-1}t^{m-1} + \cdots + A_1 t + A_0 \right] \mathrm{e}^{\alpha t}\cos(\beta t) + \\ t^s &\left[B_m t^m + B_{m-1}t^{m-1} + \cdots + B_1 t + B_0 \right] \mathrm{e}^{\alpha t}\sin(\beta t) \end{aligned}$$

where $s = 0$ if $r = \alpha \pm i\beta$ is a root of the characteristic equation and there is no match with α and β, or $s = 1$ if $r = \alpha \pm i\beta$ is a root of the characteristic equation.

Note that these two cases only cover functions $f(t)$ that can be rewritten as the forms above. If $f(t)$ cannot be expressed in the forms above, then the problem cannot be solved using the method of undetermined coefficients.

Example

$$\begin{aligned} y'' + 3y' + 2y &= 5\sin(t)t^0\mathrm{e}^0 t \\ r^2 + 3r + 2 &= (r+2)(r+1) = 0 \\ r &= -1 \quad r = -2 \\ y_p &= A\cos(t) + B\sin(t) \\ y_p' &= -A\sin(t) + B\cos(t) \\ y_p'' &= -A\cos(t) - B\sin(t) \end{aligned}$$

Since the roots are real, there is no match to α or β when forming the particular solution.

$$\begin{aligned} y'' + 3y' + 2y &= 5\sin(t) \\ -A\cos(t) - B\sin(t) + 3\Big[-A\sin(t) + B\cos(t)\Big] + 2\Big[A\cos(t) + B\sin(t)\Big] &= 5\sin(t) \\ -3A\sin(t) + B\sin(t) - A\cos(t) + 3B\cos(t) &= 5\sin(t) \\ (-3A + B)\sin(t) + (-A + 3B)\cos(t) &= 5\sin(t) \\ -3A + B = 5 \quad -A + 3B &= 0 \\ A = -\frac{3}{2} \quad B &= \frac{1}{2} \end{aligned}$$

By equating the like coefficients, we can find an explicit solution:

$$y_p = -\frac{3}{2}\cos(t) + \frac{1}{2}\sin(t)$$

Example

Find the general form of a particular solution y_p.

$$\begin{aligned} y'' + 8y' - 9y &= 10t^2 \mathrm{e}^t \cos(2t) \\ r^2 + 8r - 9 &= (r+9)(r-1) = 0 \\ r &= -9 \quad r = 1 \quad s = 0 \\ \mathrm{e}^t &= \mathrm{e}^{1t} \quad \alpha = 1 \quad \beta = 2 \\ y_p &= \Big[At^2 + Bt + C \Big] \mathrm{e}^t \cos(2t) + \Big[Dt^2 + Et + F \Big] \mathrm{e}^t \sin(2t) \end{aligned}$$

Superposition

Let y_1 be a solution to $ay''+by'+cy = f_1(t)$ and let y_2 be a solution to $ay''+by'+cy = f_2(t)$. Then $y = k_1y_1 + k_2y_2$ is a solution to $ay'' + by' + cy = k_1f_1(t) + k_2f_2(t)$. If $f_1(t) = 0$, then we can solve the homogeneous equation:

$$y_h = c_1y_2 + c_2y_2$$

and $ay'' + by' + cy = f_2(t) \neq 0$. We find a particular solution to the second equation, y_p, and the general solution to $ay'' + by' + cy = f(t)$ will be $y = y_h + y_p$, where y_h is the general solution to the corresponding homogeneous equation.

Example

Solve the following initial value problem.

$$y'' - y = -11t + 1 \quad y(0) = -1 \quad y'(0) = 1$$

We can use the method of superposition to solve this as:

$$y = y_h + y_p$$

We first need to find the solution to the corresponding homogeneous equation y_h:

$$\begin{aligned} y'' - y &= 0 \\ r^2 - 1 &= 0 \\ r &= \pm 1 \\ y_h &= c_1\mathrm{e}^t + c_2\mathrm{e}^{-t} \end{aligned}$$

Now we need to find a particular solution y_p:

$$\begin{aligned} y_p &= At + B \\ y_p' &= A \\ y_p'' &= 0 \\ 0 - (At + B) &= -11t + 1 \\ -At - B &= -11t + 1 \\ -A &= 11 \quad -B = 11 \\ y_p &= 11t - 1 \end{aligned}$$

Now we have both solutions and we can solve for the arbitrary constants:

$$
\begin{aligned}
y &= y_h + y_p = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{-t} + 11t - 1 \\
y(0) &= -1 = c_1 + c_2 - 1 \\
c_1 &= -c_2 \\
y' &= c_1 \mathrm{e}^t - c_2 \mathrm{e}^{-t} + 11 \\
y'(0) &= 1 = c_1 - c_2 + 11 \\
-1 &= -2c_2 \\
c_2 &= 5 \quad c_1 = -5 \\
y &= -5\mathrm{e}^t + 5\mathrm{e}^{-t} + 11t - 1
\end{aligned}
$$

Example

Solve the following initial value problem.

$$y'' + 9y = 27 \quad y(0) = 4 \quad y'(0) = 6$$

Homogeneous solution:

$$
\begin{aligned}
y'' + 9y &= 27 \\
y &= y_h + y_p \\
r^2 + 9 &= 0 \\
r &= 0 \pm 3i \quad \alpha = 0 \quad \beta = 3 \\
y_h &= \mathrm{e}^{0t}\Big[c_1 \cos(3t) + c_2 \sin(3t)\Big]
\end{aligned}
$$

Particular solution:

$$
\begin{aligned}
y_p &= A \\
y_p' &= y_p'' = 0 \\
0 + 9(A) &= 27 \quad A = 3 \\
y_p &= 3
\end{aligned}
$$

Solve for arbitrary constants:

$$
\begin{aligned}
y &= y_h + y_p = c_1 \cos(3t) + c_2 \sin(3t) + 3 \\
y(0) &= 4 = c_1 + 3 \\
y' &= -3c_1 \sin(3t) + 3c_2 \cos(3t) \\
y'(0) &= 6 = 3c_2 \quad c_2 = 2 \\
y &= \cos(3t) + 2\sin(3t) + 3
\end{aligned}
$$

Example

Find the general solution to the following differential equation.

$$y'' + 9y = \cos(t) + 1$$

Homogeneous solution:

$$\begin{aligned} y'' + 9y &= 0 \\ r^2 + 9 &= 0 \\ r &= 0 \pm 3i \quad \alpha = 0 \quad \beta = 3 \\ y_h &= c_1 \cos(3t) + c_2 \sin(3t) \end{aligned}$$

Particular solution:

$$\begin{aligned} y_p &= A\cos(t) + B\sin(t) + C \\ y_p' &= -A\sin(t) + B\cos(t) \\ y_p'' &= -A\cos(t) - B\sin(t) \\ -A\cos(t) - B\sin(t) + 9\Big[A\cos(t) + B\sin(t) + C\Big] &= \cos(t) + 1 \\ 8A\cos(t) + 8B\sin(t) + 9C &= \cos(t) + 1 \\ 8A = 1 \quad 8B = 0 \quad 9C &= 1 \\ A = \frac{1}{8} \quad B = 0 \quad C &= \frac{1}{9} \\ y_p &= \frac{\cos(t)}{8} + \frac{1}{9} \end{aligned}$$

General solution:

$$y = c_1 \cos(3t) + c_2 \sin(3t) + \frac{\cos(t)}{8} + \frac{1}{9}$$

Example

Solve the following initial value problem.

$$y'' + 4y = \sin(t) - \cos(t) \quad y(0) = 1 \quad y'(0) = 0$$

Homogeneous solution:

$$\begin{aligned} y'' + 4y &= 0 \\ r^2 + 4 &= 0 \\ r &= 0 \pm 2i \quad \alpha = 0 \quad \beta = 2 \end{aligned}$$

Particular solution:

$$\begin{aligned}
y_p &= A\cos(t) + B\sin(t) \\
y_p' &= -A\sin(t) + B\cos(t) \\
y_p'' &= -A\cos(t) - B\sin(t) \\
-A\cos(t) - B\sin(t) + 4\Big[A\cos(t) + B\sin(t)\Big] &= \sin(t) - \cos(t) \\
3A\cos(t) + 3B\sin(t) &= \sin(t) - \cos(t) \\
3A &= -1 \quad A = -\frac{1}{3} \\
3B &= 1 \quad B = \frac{1}{3} \\
y_p &= -\frac{1}{3}\cos(t) + \frac{1}{3}\sin(t)
\end{aligned}$$

Solve for arbitrary constants:

$$\begin{aligned}
y &= y_h + y_p \\
y &= c_1\cos(2t) + c_2\sin(2t) - \frac{1}{3}\cos(t) + \frac{1}{3}\sin(t) \\
y(0) &= 1 = c_1 - \frac{1}{3} \\
c_1 &= \frac{4}{3} \\
y' &= -2c_1\sin(2t) + 2c_2\cos(2t) + \frac{1}{3}\sin(t) + \frac{1}{3}\cos(t) \\
y'(0) &= 0 = 2c_2 + \frac{1}{3} \\
c_2 &= -\frac{1}{6} \\
y &= \frac{4}{3}\cos(2t) - \frac{1}{6}\sin(2t) - \frac{1}{3}\cos(t) + \frac{1}{3}\sin(t)
\end{aligned}$$

Variation of Parameters

This technique is used to find a particular solution, y_p, to

$$ay'' + by' + cy = f(t) \neq 0$$

if the method of undetermined coefficients is not applicable. Given this equation:

$$\begin{aligned}
y &= y_h + y_p \\
y_h &= c_1 y_1 + c_2 y_2 \\
y_p &= v_1 y_1 + v_2 y_2 \\
v_1 &= -\int \frac{f(t) y_2}{aW[y_1, y_2]}\, \mathrm{d}t \\
v_2 &= \int \frac{f(t) y_1}{aW[y_1, y_2]}\, \mathrm{d}t
\end{aligned}$$

Example

Find the general solution to the following differential equation.

$$y'' + 2y' + y = \mathrm{e}^{-t} \ln(t)$$

Homogeneous Solution:

$$\begin{aligned}
r^2 + 2r + 1 &= 0 \\
(r+1)^2 &= 0 \\
r &= -1 \\
y_h &= c_1 \mathrm{e}^{-t} + c_2 t\mathrm{e}^{-t}
\end{aligned}$$

Particular Solution:

$$\begin{aligned}
y_h &= c_1 \mathrm{e}^{-t} + c_2 t\mathrm{e}^{-t} \\
y_p &= v_1 y_1 + v_2 y_2 \\
y_1 &= \mathrm{e}^{-t} \quad y_2 = t\mathrm{e}^{-t} \\
v_1 &= -\int \frac{\mathrm{e}^{-t} \ln(t) t\mathrm{e}^{-t}}{W[\mathrm{e}^{-t}, t\mathrm{e}^{-t}]}\, \mathrm{d}t \\
W[\mathrm{e}^{-t}, t\mathrm{e}^{t}] &= \begin{vmatrix} \mathrm{e}^{-t} & t\mathrm{e}^{-t} \\ -\mathrm{e}^{-t} & \mathrm{e}^{-t} - t\mathrm{e}^{-t} \end{vmatrix} = \mathrm{e}^{-2t} \\
v_1 &= -\int \frac{\mathrm{e}^{-2t} t \ln(t)}{\mathrm{e}^{-2t}}\, \mathrm{d}t = \frac{t^2}{4} - \frac{t^2}{2} \ln(t)
\end{aligned}$$

$$
\begin{aligned}
v_2 &= \int \frac{\mathrm{e}^{-t}\ln(t)}{\mathrm{e}^{-2t}} \\
&= \int \ln(t)\ \mathrm{d}t \\
&= t(\ln(t)-1) \\
y &= y_h + y_p \\
y &= c_1\mathrm{e}^{-t} + c_2 t\mathrm{e}^{-t} + \left[\frac{t^2}{4} - \frac{t^2}{2}\ln(t)\right]\mathrm{e}^{-t} + \Big[t(\ln(t)-1)\Big]t\mathrm{e}^{-t}
\end{aligned}
$$

Example

Find the general solution to the following differential equation over $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$$y'' + y = \tan(t)$$

Homogeneous Solution:

$$
\begin{aligned}
r^2 + 1 &= 0 \\
r^2 &= -1 \\
r &= 0 \pm i \quad \alpha = 0 \quad \beta = 1 \\
y_h &= c_1\cos(t) + c_2\sin(t)
\end{aligned}
$$

Particular Solution:

$$
\begin{aligned}
y_h &= c_1\cos(t) + c_2\sin(t) \\
y_p &= v_1y_1 + v_2y_2 \\
v_1 &= -\int \frac{\tan(t)\sin(t)}{W[\cos(t),\sin(t)]}\ \mathrm{d}t \\
W[\cos(t),\sin(t)] &= 1 \\
v_1 &= -\int \frac{\sin(t)\sin(t)}{\cos(t)}\ \mathrm{d}t = -\int \frac{\sin^2(t)}{\cos(t)}\ \mathrm{d}t \\
&= -\int \frac{1-\cos^2(t)}{\cos(t)}\ \mathrm{d}t \\
&= -\left[\int \frac{1}{\cos(t)}\ \mathrm{d}t - \int \cos(t)\ \mathrm{d}t\right] \\
&= \sin(t) - \ln|\sec(t) + \tan(t)|
\end{aligned}
$$

$$
\begin{aligned}
v_2 &= \int \tan(t)\cos(t)\ \mathrm{d}t \\
&= \int \sin(t)\ \mathrm{d}t \\
&= -\cos(t) \\
y &= y_h + y_p \\
y &= c_1\cos(t) + c_2\sin(t) + \Big[\sin(t) - \ln|\sec(t) + \tan(t)|\Big]\cos(t) - \cos(t)\sin(t)
\end{aligned}
$$

Mass Spring Systems

Free Mechanical Vibrations

The governing equation for mass spring systems is given by

$$my'' + by' + ky = f(t)$$

where m is mass in kilograms, b is a damping coefficient (friction) in Newton-seconds per meter, k is the spring constant (stiffness) in Newtons per meter, and f is the external force. For free mechanical vibrations, $f(t) = 0$.

Simple Harmonic Motion

When we deal with simple harmonic motion, there is no external force, and friction is negligible. Thus, our equation becomes:

$$my'' + ky = 0$$

We can solve this equation using the techniques above.

$$\begin{aligned}
y'' + \frac{k}{m}y &= 0 \\
Let: \omega &= \sqrt{\frac{k}{m}} \\
\omega^2 &= \frac{k}{m} \\
y'' + \omega^2 y &= 0 \\
r^2 + \omega^2 &= 0 \\
r^2 &= -\omega^2 \\
r &= 0 \pm \omega i \quad \alpha = 0 \quad \beta = \omega \\
y &= c_1 \cos(\omega t) + c_2 \sin(\omega t)
\end{aligned}$$

Given an initial value problem, we can solve for c_1 and c_2. We can express y as the following equation:

$$y = A\sin(\omega t + \phi)$$

where $A = \sqrt{c_1^2 + c_2^2}$ is the amplitude, ϕ is a phase angle, ω is the angular frequency in radians per second, $T = \frac{2\pi}{\omega}$ is the period in seconds, and $\frac{\omega}{2\pi}$ is the natural frequency in cycles per second. To find ϕ and A in terms of c_1 and c_2, consider the following trigonometric identity:

$$\begin{aligned}
\sin(\alpha+\beta) &= \cos\alpha\sin\beta + \cos\beta\sin\alpha \\
\sin(\omega t+\phi) &= \cos(\omega t)\sin\phi + \cos\phi\sin(\omega t) \\
A\Bigg[\sin(\omega t+\phi)\Bigg] &= A\cos(\omega t)\sin\phi + A\cos\phi\sin(\omega t) \\
y = A\sin(\omega t+\phi) &= c_1\cos(\omega t) + c_2\sin(\omega t) \\
A\sin\phi &= c_1 \quad c_1 = A\sin\phi \\
A\cos\phi &= c_2 \quad c_2 = A\cos\phi \\
A &= \sqrt{c_1^2+c_2^2} \\
\tan\phi &= \frac{c_1}{c_2} \quad \phi = \tan^{-1}(\frac{c_1}{c_2})
\end{aligned}$$

The signs of c_1 and c_2 will determine what quadrant ϕ lies in.

Example

A 1kg mass is attached to a spring with stiffness $k = 64\frac{N}{m}$. The mass is stretched in the positive direction from the equilibrium position $y = 0$ by $\frac{2}{3}m$, and given an initial velocity of $-\frac{4}{3}\frac{m}{s}$ in the opposite direction. Neglecting external forces and friction:

1. Find the equation of motion.

$$\begin{aligned}
y'' + 64y &= 0 \\
y(0) &= \frac{2}{3} \quad y'(0) = -\frac{4}{3} \\
r^2 + 64 &= 0 \quad r = 0 \pm 8i \\
y(t) &= c_1\cos(8t) + c_2\sin(8t) \\
y'(t) &= -8c_1\sin(8t) + 8c_2\cos(8t)
\end{aligned}$$

$$
\begin{aligned}
y(0) &= \frac{2}{3} = c_1 \\
y'(0) &= -\frac{4}{3} = 8c_2 \\
y &= \frac{2}{3}\cos(8t) - \frac{1}{6}\sin(8t) \\
A &= \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{6}\right)^2} = \frac{\sqrt{17}}{6} \\
\tan\phi &= \frac{c_1}{c_2} = -4
\end{aligned}
$$

From the signs of c_1 and c_2, we know ϕ lies in quadrant II.

$$
\begin{aligned}
\phi = \tan^{-1}(-4) &\approx (-1.326 + \pi)\ rad \approx 1.816\ rad \\
y(t) &= \frac{\sqrt{17}}{6}\sin(8t + 1.816)
\end{aligned}
$$

2. Find the first time that the mass passes over the equilibrium position.

$$
\begin{aligned}
y(t) = 0 &= \frac{\sqrt{17}}{6}\sin(8t + 1.816) \\
\sin(8t + 1.816) &= 0 \\
8t + 1.816 &= n\pi, \quad n \in \mathbb{Z} \\
8t &= n\pi - 1.816 \\
t &= \frac{n\pi - 1.816}{8} \\
t &= \frac{\pi - 1.816}{8}
\end{aligned}
$$

Under-damped Motion

When the damping coefficient exists but is small relative to the spring's stiffness, oscillation occurs and eventually flattens out. The roots of the characteristic equation will typically be complex.

Example

A 1kg mass is attached to a spring with constant $k = 10\frac{N}{m}$ and a damping coefficient of $b = 2\frac{N \cdot s}{m}$. Find the equation of motion given the initial conditions $y(0) = -1$ and

$y'(0) = 0$.

$$my'' + by' + ky = 0$$
$$y'' + 2y' + 10 = 0$$
$$r = \frac{-2 \pm \sqrt{4 - (4)(10)}}{2}$$
$$= -1 \pm 3i \quad \alpha = -1 \quad \beta = 3$$
$$y = \mathrm{e}^{-t}\Bigg[c_1 \cos(3t) + c_2 \sin(3t)\Bigg]$$
$$y' = \mathrm{e}^{-t}\Bigg[-3c_1 \sin(3t) + 3c_2 \cos(3t)\Bigg] - \mathrm{e}^{-t}\Bigg[c_1 \cos(3t) + c_2 \sin(3t)\Bigg]$$
$$y(0) = -1 = c_1$$
$$y'(0) = 0 = 3c_2 - c_1$$
$$c_2 = -\frac{1}{3}$$
$$y = \mathrm{e}^{-t}\Bigg[-\cos(3t) - \frac{1}{3}\sin(3t)\Bigg]$$

Now we need to rewrite this in the form $y = A\mathrm{e}^{-t}\Bigg[\sin(3t + \phi)\Bigg]$:

$$A = \sqrt{(-1)^2 + (-\frac{1}{3})^2} = \frac{\sqrt{10}}{3}$$
$$\tan\phi = \frac{c_1}{c_2} = 3$$

From the signs of c_1 and c_2, we know ϕ lies in quadrant III.

$$\phi = \tan^{-1}(-4) \approx (1.25 + \pi)\ rad \approx 4.39\ rad$$
$$y(t) = \frac{\sqrt{10}}{3}\mathrm{e}^{-t}\Bigg[\sin(3t + 4.39)\Bigg]$$

Note the following property of this equation of motion:

$$\lim_{t\to\infty} y(t) = 0$$

This is due the the amplitude also being a variable function of time that approaches 0 as time goes to infinity. This is not a periodic function. A "quasi-period" $\frac{2\pi}{\beta} = \frac{2\pi}{3}$ exists, as well as a "quasi-frequency" $\frac{\beta}{2\pi} = \frac{3}{2\pi}$.

Over-damped Motion

This can occur when the damping coefficient is large compared to the spring stiffness. The roots of the characteristic equation will typically be real and distinct.

Example

Find the equation of motion for $y'' + 3y' + 2y = 0$ given the initial conditions $y(0) = 1$ and $y'(0) = 0$.

$$\begin{aligned}
y'' + 3y' + 2y &= 0 \\
r^2 + 3r + 2 &= (r+1)(r+2) = 0 \\
r &= -1 \quad r = -2 \\
y(t) &= c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-2t} \\
y'(t) &= -c_1 \mathrm{e}^{-t} - 2c_2 \mathrm{e}^{-t} \\
y(0) &= 1 = c_1 + c_2 \quad c_1 = 1 - c_2 \\
y'(0) &= 0 = -c_1 - c_2 \\
c_1 &= 2 \quad c_2 = -1 \\
y(t) &= 2\mathrm{e}^{-t} - \mathrm{e}^{-2t}
\end{aligned}$$

Notice here that $\lim_{t\to\infty} y(t) = 0$ and there is no oscillation.

Critically-damped Systems

Suppose we a spring system defined by $y'' + 10y' + 25y = 0$ with the initial conditions $y(0) = 1$ and $y'(0) = 0$.

$$\begin{aligned}
y'' + 10y' + 25y &= 0 \\
r^2 + 10r + 25 &= (r+5)^2 = 0 \\
r &= -5 \\
y(t) &= c_1 \mathrm{e}^{-5t} + c_2 t \mathrm{e}^{-5t} \\
y'(t) &= -5c_1 \mathrm{e}^{-5t} + c_2 \mathrm{e}^{-5t} - 5c_2 t \mathrm{e}^{-5t} \\
y(0) &= 1 = c_1 \\
y'(0) &= 0 = -5 + c_2 \quad c_2 = 5 \\
y(t) &= \mathrm{e}^{-5t} + 5t\mathrm{e}^{-5t}
\end{aligned}$$

Note that this system also will not oscillate as $\lim_{t\to\infty} y(t) = 0$.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech