Differential Equations

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Separable Equations

Consider:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p(y)g(x)$$

In this case, we can rewrite it as:

$$\frac{\mathrm{d}y}{p(y)} = g(x)\ \mathrm{d}x$$

If we let $h(y) = \frac{1}{p(y)}$:

$$h(y)\ \mathrm{d}y = g(x)\ \mathrm{d}x$$
$$\int h(y)\ \mathrm{d}y = \int g(x)\ \mathrm{d}x$$
$$H(y) = G(x) + c$$

This yields an implicit solution. Separable equations can be linear or non-linear.

Example

Solve the following:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 8x^3\mathrm{e}^{-2y}$$

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 8x^3\mathrm{e}^{-2y} \\
\frac{\mathrm{d}y}{\mathrm{e}^{-2y}} &= 8x^3\ \mathrm{d}x \\
\mathrm{e}^{2y}\ \mathrm{d}y &= 8x^3\ \mathrm{d}x \\
\int \mathrm{e}^{2y}\ \mathrm{d}y &= \int 8x^3\ \mathrm{d}x \\
\frac{1}{2}\mathrm{e}^{2y} + c_1 &= 2x^4 + c_2 \\
\frac{1}{2}\mathrm{e}^{2y} &= 2x^4 + c
\end{aligned}
$$

We can also solve this explicitly:

$$
\begin{aligned}
\frac{1}{2}\mathrm{e}^{2y} &= 2x^4 + c \\
\ln(\mathrm{e}^{2y}) &= \ln(4x^4 + c) \\
2y &= \ln(4x^4 + c) \\
y &= \frac{\ln(4x^4 + c)}{2}
\end{aligned}
$$

Example

Solve the following initial value problem:

$$
\frac{\mathrm{d}y}{\mathrm{d}x} = (1 + y^2)\tan(x) \quad y(0) = \sqrt{3}
$$

This is a non-linear separable equation.

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= (1 + y^2)\tan(x) \\
\frac{\mathrm{d}y}{1 + y^2} &= \tan(x)\ \mathrm{d}x \\
\int \frac{\mathrm{d}y}{1 + y^2} &= \int \tan(x)\ \mathrm{d}x \\
\tan^{-1}(y) &= \ln|\sec(x)| + c
\end{aligned}
$$

Using our initial value $y(0) = \sqrt{3}$:

$$\begin{aligned}\tan^{-1}(\sqrt{3}) &= \ln|\sec(0)| + c\\ \tan^{-1}(\sqrt{3}) &= \ln|1| + c\\ \tan^{-1}(\sqrt{3}) &= c\\ c &= \frac{\pi}{3} \text{ over } \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\end{aligned}$$

We can now solve explicitly for y:

$$\begin{aligned}\tan^{-1}(y) &= \ln|\sec(x)| + \frac{\pi}{3}\\ \tan(\tan^{-1}(y)) &= \tan\left(\ln|sec(x)| + \frac{\pi}{3}\right)\\ y &= \tan\left(\ln|sec(x)| + \frac{\pi}{3}\right)\end{aligned}$$

Example

Solve the following initial value problem over $[0, \infty)$:

$$\sqrt{y}\ \mathrm{d}x + (1+x)\ \mathrm{d}y = 0 \quad y(0) = 1$$

$$\begin{aligned}\sqrt{y}\ \mathrm{d}x + (1+x)\ \mathrm{d}y &= 0\\ (1+x)\ \mathrm{d}y &= -\sqrt{y}\ \mathrm{d}x\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= (-\sqrt{y})\frac{1}{1+x}\\ \frac{\mathrm{d}y}{\sqrt{y}} &= -\frac{\mathrm{d}x}{1+x}\\ \int y^{-\frac{1}{2}}\ \mathrm{d}y &= -\int \frac{\mathrm{d}x}{1+x}\\ 2y^{\frac{1}{2}} &= -\ln|1+x| + c\end{aligned}$$

Using our initial value $y(0) = 1$:

$$\begin{aligned}2 &= -\ln|1| + c\\ c &= 2\end{aligned}$$

We can now solve explicitly for y:

$$\begin{aligned} 2\sqrt{y} &= 2 - \ln|1+x| \\ \sqrt{y} &= \frac{2 - \ln|1+x|}{2} \\ y &= \left[\frac{2 - \ln|1+x|}{2}\right]^2 \end{aligned}$$

Example

Solve the initial value problem:

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = y - xy \quad y(1) = 1$$

$$\begin{aligned} x^2 \frac{\mathrm{d}y}{\mathrm{d}x} &= y - xy \\ x^2 \frac{\mathrm{d}y}{\mathrm{d}x} &= y(1-x) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= (y)\left(\frac{1-x}{x^2}\right) \\ \frac{\mathrm{d}y}{y} &= \frac{1-x}{x^2}\ \mathrm{d}x \\ \int \frac{\mathrm{d}y}{y} &= \int \left[\frac{1}{x^2} - \frac{1}{x}\right]\ \mathrm{d}x \\ \ln|y| &= -\frac{1}{x} - \ln|x| + c \\ \mathrm{e}^{\ln|y|} &= \mathrm{e}^{-\frac{1}{x} - \ln|x| + c} \\ |y| &= \mathrm{e}^{-\frac{1}{x}}\mathrm{e}^{-\ln|x|}\mathrm{e}^{c} \\ &= \mathrm{e}^{-\frac{1}{x}}\frac{1}{|x|}\mathrm{e}^{c} \\ Let: k &= \pm\mathrm{e}^{c} \\ y &= \frac{k\mathrm{e}^{-\frac{1}{x}}}{x} \end{aligned}$$

Using our initial value $y(1) = 1$:

$$\begin{aligned} 1 &= \frac{k\mathrm{e}^{-1}}{1} \\ k &= \mathrm{e} \end{aligned}$$

We can now solve explicitly for y:

$$\begin{aligned} y &= \frac{k\mathrm{e}^{-\frac{1}{x}}}{x} \\ y &= \frac{\mathrm{e}^{1-\frac{1}{x}}}{x} \end{aligned}$$

Example

Solve the initial value problem:

$$\frac{1}{\theta}\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{y\sin\theta}{y^2+1} \quad y(\pi) = 1$$

$$\begin{aligned} \frac{1}{\theta}\frac{\mathrm{d}y}{\mathrm{d}\theta} &= \frac{y\sin\theta}{y^2+1} \\ \frac{\mathrm{d}y}{\mathrm{d}\theta} &= (\theta\sin\theta)(\frac{y}{y^2+1}) \\ \frac{y^2+1}{y}\ \mathrm{d}y &= \theta\sin\theta\ \mathrm{d}\theta \\ \int \frac{y^2+1}{y}\ \mathrm{d}y &= \int \theta\sin\theta\ \mathrm{d}\theta \\ \frac{y^2}{2} + \ln|y| + c &= -\theta\cos\theta - \int(-\cos\theta\ \mathrm{d}\theta) \\ &= -\theta\cos\theta + \sin\theta + c \end{aligned}$$

Using our initial value $y(\pi) = 1$:

$$\begin{aligned} \frac{1}{2} &= -\pi\cos(\pi) + \sin(\pi) + c \\ \frac{1}{2} &= -\pi(-1) + c \\ c &= \frac{1}{2} - \pi \end{aligned}$$

We can now solve explicitly for y:

$$\frac{y^2}{2} + \ln|y| = -\theta\cos\theta + \sin\theta + (\frac{1}{2} - \pi)$$

First Order Linear Equations

General form:

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

Let's consider:

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^2$$

By the product rule we can see that the left hand side is equal to $\frac{\mathrm{d}y}{\mathrm{d}x}\Big[xy\Big]$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[xy\Big] = x^2$$

Integrating both sides:

$$\begin{aligned}\int \frac{\mathrm{d}}{\mathrm{d}x}\Big[xy\Big]\ \mathrm{d}x &= \int x^2\ \mathrm{d}x \\ xy &= \frac{x^3}{3} + c \\ y &= \frac{x^2}{3} + \frac{c}{x}\end{aligned}$$

Standard Form

Consider:

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

Write it in standard form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

Let $P(x) = \frac{a_0(x)}{a_1(x)}$ and $Q(x) = \frac{g(x)}{a_1(x)}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

This is a first order linear equation in standard form. In order to solve this, we need to determine a standard function of x called μ. We multiply the equation by μ:

$$\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \mu(x)P(x)y = \mu(x)Q(x)$$

We require that that $\mu'(x) = \mu(x)P(x)$. To find this function μ:

$$\begin{aligned}
\frac{\mu'(x)}{\mu(x)} &= P(x) \\
\int \frac{\mu'(x)}{\mu(x)} &= \mu P(x) \\
\ln|\mu(x)| &= \int P(x)\ \mathrm{d}x + c \\
c &= 0 \\
\mathrm{e}^{\ln|\mu|} &= \mathrm{e}^{\int P(x)\ \mathrm{d}x} \\
|\mu(x)| &= \mathrm{e}^{\int P(x)\ \mathrm{d}x}
\end{aligned}$$

This is called an integrating factor. From this, we can determine that:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\Big[\mu(x)y\Big] &= \mu(x)Q(x) \\
\int \frac{\mathrm{d}}{\mathrm{d}x}\Big[\mu(x)y\Big] &= \int \mu(x)Q(x) \\
\mu(x)y &= \int \mu(x)Q(x)\ \mathrm{d}x \\
y &= \frac{1}{\mu(x)}\Big[\mu(x)Q(x)\ \mathrm{d}x\Big] \\
\mu(x) &= \mathrm{e}^{\int P(x)\ \mathrm{d}x}
\end{aligned}$$

Example

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 6$$

This is a first order linear equation already in standard form. It is also separable, but we will solve using the method for first order linear equations.

1. Identify $P(x)$:

$$P(x) = -3 \quad Q(x) = 6$$

2. Find $\mu(x)$:

$$\mu(x) = \mathrm{e}^{\int -3\ \mathrm{d}x} = \mathrm{e}^{-3x}$$

3. Multiply the given equation by $\mu(x)$:

$$\begin{aligned}
\mathrm{e}^{-3x}\frac{\mathrm{d}y}{\mathrm{d}x} - 3\mathrm{e}^{-3x}y &= 6\mathrm{e}^{-3x} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-3x}y\right] &= 6\mathrm{e}^{-3x} \\
\int \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-3x}y\right]\,\mathrm{d}x &= \int 6\mathrm{e}^{-3x}\,\mathrm{d}x \\
\mathrm{e}^{-3x}y &= 6(-\frac{1}{3}\mathrm{e}^{-3x}) + c \\
&= -2\mathrm{e}^{-3x} + c \\
y &= -2 + C\mathrm{e}^{3x}
\end{aligned}$$

Example

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = x^6\mathrm{e}^x$$

1. Write it in standard form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{4}{x}y = x^5\mathrm{e}^x$$

2. Identify $P(x)$:

$$P(x) = \frac{-4}{x}$$

3. Find $\mu(x)$:

$$\mu(x) = \mathrm{e}^{\int \frac{-4}{x}\,\mathrm{d}x} = \mathrm{e}^{-4\ln|x|} = \mathrm{e}^{\ln\frac{1}{x^4}} = \frac{1}{x^4}$$

4. Multiply the given equation by $\mu(x)$:

$$\begin{aligned}
\frac{1}{x^4}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{4}{x^5}y &= x\mathrm{e}^x \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x^4}y\right] &= x\mathrm{e}^x \\
\frac{1}{x^4}y &= \int x\mathrm{e}^x\,\mathrm{d}x = x\mathrm{e}^x - \int \mathrm{e}^x\,\mathrm{d}x = x\mathrm{e}^x - \mathrm{e}^x + c \\
y &= x^5\mathrm{e}^x - x^4\mathrm{e}^x + Cx^4
\end{aligned}$$

Example

Solve the following over $(0, \infty)$:

$$\begin{aligned}
x\frac{\mathrm{d}y}{\mathrm{d}x} + 3(y + x^2) = \frac{\cos(x)}{x^2} \\
x\frac{\mathrm{d}y}{\mathrm{d}x} + 3y &= \frac{\cos(x)}{x^2} - 3x^2 \\
\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{3}{x}y &= \frac{\cos(x)}{x^3} - 3x \\
P(x) &= \frac{3}{x} \\
\mu(x) &= \mathrm{e}^{\int \frac{3}{x}\ \mathrm{d}x} = \mathrm{e}^{3\ln(x)} = \mathrm{e}^{\ln(x^3)} = x^3 \\
x^3\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2y &= \cos(x) + 3x^4 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[x^3y\right] &= \cos(x) + 3x^4 \\
\int \frac{\mathrm{d}}{\mathrm{d}x}\left[x^3y\right]\ \mathrm{d}x &= \int \cos(x) + 3x^4\ \mathrm{d}x \\
x^3y &= \sin(x) + \frac{3x^5}{5} + c \\
y &= \frac{\sin(x)}{x^3} - \frac{3x^2}{5} + \frac{c}{x^3}
\end{aligned}$$

Example

Solve the following initial value problem over $(-\frac{\pi}{2}, \frac{\pi}{2})$ given $y(0) = 2$:

$$\begin{aligned}
\cos(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \sin(x)y = 1 \\
\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\sin(x)}{\cos(x)}y &= \frac{1}{\cos(x)} \\
\frac{\mathrm{d}y}{\mathrm{d}x} + \tan(x)y &= \sec(x) \\
\mu(x) &= \mathrm{e}^{\int P(x)\ \mathrm{d}x} = \mathrm{e}^{\int \tan(x)\ \mathrm{d}x} = \mathrm{e}^{\ln|\sec(x)|} = \sec(x) \\
\sec(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \sec(x)\tan(x)y &= \sec^2(x)
\end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[\sec(x)y\right] &= \sec^2(x) \\
\int \frac{\mathrm{d}}{\mathrm{d}x}\left[\sec(x)y\right]\ \mathrm{d}x &= \int \sec^2(x)\ \mathrm{d}x \\
\sec(x)y &= \tan(x) + c \\
y &= \frac{\tan(x)}{\sec(x)} + \frac{c}{\sec(x)} \\
&= \sin(x) + c\cos(x)
\end{aligned}
$$

Using our initial value:

$$
\begin{aligned}
y &= \sin(x) + c\cos(x) \\
2 &= \sin(0) + c\cos(0) \\
&= 0 + c \\
c &= 2 \\
y &= \sin(x) + 2\cos(x)
\end{aligned}
$$

Substitutions and Transformations

Consider:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

where n is any real number. This is called a **Bernoulli Equation**. We use the substitution $v = y^{1-n}$ to transform the equation into a new variable:

$$\frac{\mathrm{d}v}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}$$

This gives a linear equation in v:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1-n}y^n\frac{\mathrm{d}v}{\mathrm{d}x}$$

We divide the equation by y^n:

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y(y^{-n}) = Q(x)$$

We can now substitute for $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\begin{aligned}
\frac{1}{1-n}y^n\frac{\mathrm{d}v}{\mathrm{d}x}y^{-n} + P(x)y^{1-n} &= Q(x)\\
\frac{1}{1-n}\frac{\mathrm{d}v}{\mathrm{d}x} + P(x)v &= Q(x)\\
\frac{\mathrm{d}v}{\mathrm{d}x} + (1-n)P(x)v &= (1-n)Q(x)
\end{aligned}$$

Using this substitution allows us to change our equation into a first-order linear equation in standard form, which we can now solve.

Example

Solve the following over $(0, \infty)$:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} &= x^2 y^2 \\
\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x} y &= x^2 y^2 \\
v &= y^{1-2} = y^{-1} \\
\frac{\mathrm{d}v}{\mathrm{d}x} &= -y^{-2} \frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -y^2 \frac{\mathrm{d}v}{\mathrm{d}x} \\
\frac{1}{y^2}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x} y = x^2 y^2 \right) \\
y^{-2} \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x} y(y^{-2}) &= x^2 \\
y^{-2}(-y^2 \frac{\mathrm{d}v}{\mathrm{d}x}) + \frac{1}{x} y^{-1} &= x^2 \\
-\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{1}{x} v &= x^2 \\
\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{1}{x} v &= -x^2
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{1}{x} v &= -x^2 \\
\mu(x) &= \mathrm{e}^{\int -\frac{\mathrm{d}x}{x}} = \mathrm{e}^{-\ln|x|} = \mathrm{e}^{\ln|\frac{1}{x}|} = \frac{1}{x} \\
\frac{1}{x} \frac{\mathrm{d}v}{\mathrm{d}x} - \frac{1}{x^2} v &= -x \\
\int \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x} v\right] \mathrm{d}x &= -\int x \; \mathrm{d}x \\
\frac{1}{x} v &= -\frac{x^2}{2} + c \\
v &= -\frac{x^3}{2} + cx \\
\frac{1}{y} &= -\frac{x^3}{2} + cx
\end{aligned}$$

Example

Solve the following initial value problem given $y(1) = 1$:

$$\begin{aligned} x^2 \frac{\mathrm{d}y}{\mathrm{d}x} - 2xy &= 3y^4 \\ \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x}y &= \frac{3y^4}{x^2} \\ n &= 4 \quad v = y^{1-4} = y^{-3} \\ \frac{\mathrm{d}v}{\mathrm{d}x} &= -3y^{-4}\frac{\mathrm{d}y}{\mathrm{d}x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{1}{3}y^4\frac{\mathrm{d}v}{\mathrm{d}x} \\ \frac{1}{y^4}\left(\frac{\mathrm{d}y}{\mathrm{d}x} \right. &= \left. -\frac{1}{3}y^4\frac{\mathrm{d}v}{\mathrm{d}x}\right) \\ y^{-4}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x}y(y^{-4}) &= \frac{3}{x^2} \\ y^{-4}(-\frac{1}{3}y^4\frac{\mathrm{d}v}{\mathrm{d}x}) - \frac{2}{x}v &= \frac{3}{x^2} \\ -\frac{1}{3}\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{2}{x}v &= \frac{3}{x^2} \\ \frac{\mathrm{d}v}{\mathrm{d}x} + \frac{6}{x}v &= -\frac{9}{x^2} \end{aligned}$$

This equation is now a first order linear equation in v.

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{6}{x}v &= -\frac{9}{x^2} \\
\mu(x) &= \mathrm{e}^{\int \frac{6}{x}\ \mathrm{d}x} = \mathrm{e}^{6\ln|x|} = \mathrm{e}^{\ln|x^6|} = x^6 \\
x^6\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{6x^5}{v} &= -9x^4 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[x^6 v\right] &= -9x^4 \\
\int \frac{\mathrm{d}}{\mathrm{d}x}\left[x^6 v\right]\ \mathrm{d}x &= \int -9x^4\ \mathrm{d}x \\
x^6 v &= -\frac{9}{5}x^5 + c \\
v &= -\frac{9}{5x} + \frac{c}{x^6} \\
\frac{1}{y^3} &= -\frac{9}{5x} + \frac{c}{x^6}
\end{aligned}$$

Using our initial value:

$$\begin{aligned}
\frac{1}{y^3} &= -\frac{9}{5x} + \frac{c}{x^6} \\
\frac{1}{1^3} &= -\frac{9}{5} + \frac{c}{1^6} \\
c &= \frac{14}{5} \\
\frac{1}{y^3} &= -\frac{9}{5}\frac{1}{x} + \frac{14}{5}\frac{1}{x^6}
\end{aligned}$$

Exact Equations

Consider:

$$z = f(x, y) \quad \text{vs} \quad y = f(x)$$

$z = f(x, y)$ is a surface in 3D, while $y = f(x)$ is a curve in 2D. Recall that when differentiating $y = f(x)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$ and $\mathrm{d}y = f'(x)\ \mathrm{d}x$. To differentiate $z = f(x, y)$ we need to find partial derivatives. These are computed by treating y or x as a constant and taking the derivative with respect to other variable.

$$\begin{aligned}
x = f(x, y) &= x^3y^4 - 2x^3y^5 \\
f_x = \frac{\partial f}{\partial x} &= 3x^2y^4 - 6x^2y^5 \\
f_y = \frac{\partial f}{\partial y} &= 4x^3y^3 - 10x^3y^4 \\
f_{xy} = \frac{\partial^2 f}{\partial y \partial x} &= 12x^2y^3 - 30x^2y^4 \\
f_{yx} = \frac{\partial^2 f}{\partial y \partial x} &= 12x^2y^3 - 30x^2y^4
\end{aligned}$$

We can also integrate $f(x, y)$ with respect to x or with respect to y.

$$\int xy^3\ \mathrm{d}y = x \int y^2\ \mathrm{d}y = \frac{xy^3}{3} + c$$

$\mathrm{d}y$ tells us the variable of integration, with x being constant with respect to y.

$$\int xy^2\ \mathrm{d}x = \frac{x^2y^2}{2} + c$$

$\mathrm{d}x$ tells us the variable of integration, with y being constant with respect to x. Consider again $z = f(x, y)$, the total differential is:

$$\mathrm{d}z = \frac{\partial f}{\partial x}\ \mathrm{d}x + \frac{\partial f}{\partial y}\ \mathrm{d}y$$

This is a first order equation. This equation is called an exact equation if:

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$$

We can write this as:

$$M(x,y)\ \mathrm{d}x + N(x,y)\ \mathrm{d}y = 0$$

where $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$. We want to find $f(x,y) = c$, geometrically represented as a level curve, or a slice of the curve $z = f(x,y)$. Initially:

$$f(x,y) = \int M(x,y)\ \mathrm{d}x + h(y)$$

We need to find $h(y)$ by differentiating with respect to y:

$$\frac{\partial}{\partial y}\left[\int M(x,y)\ \mathrm{d}y\right] = h'(y)$$

$$\int h'(y)\ \mathrm{d}y = h(y) + c$$

$$f(x,y) = \int M(x,y)\ \mathrm{d}x + h(y) + c = 0$$

Example

Solve the initial value problem given $y(1) = 1$:

$$(x^2y^3)\ \mathrm{d}x + (x^3y^2)\ \mathrm{d}y = 0$$

$$M(x,y) = x^2y^3 \quad N(x,y) = x^3y^2$$

$$\frac{\partial M}{\partial y} = 3x^2y^2 \quad \frac{\partial N}{\partial x} = 3x^2y^2$$

From this, we can determine that it is an exact equation.

$$\begin{aligned}
f(x,y) &= \int x^2y^3\ \mathrm{d}x + h(y) \\
&= \frac{x^3y^3}{3} + h(y) \\
\frac{\partial f}{\partial y} &= x^3y^2 + h'(y) = N(x,y) = x^3y^2 \\
\therefore h'(y) &= 0 \\
h(y) &= k \\
f(x,y) &= c = \frac{x^3y^3}{3} + k \\
\frac{x^3y^3}{3} &= c
\end{aligned}$$

Using our initial value:

$$\frac{x^3y^3}{3} = c$$
$$\frac{(1)(1)}{3} = c$$
$$\frac{x^3y^3}{3} = \frac{1}{3}$$

Example

$$(\mathrm{e}^{2y} - y\cos(xy))\ \mathrm{d}x + (2x\mathrm{e}^{2y} - x\cos(xy) + 2y)\ \mathrm{d}y = 0$$
$$M(x,y) = \mathrm{e}^{2y} - y\cos(xy) \quad N(x,y) = 2x\mathrm{e}^{2y} - x\cos(xy) + 2y$$
$$\frac{\partial M}{\partial y} = 2\mathrm{e}^{2y} - (\cos(xy) + y(-\sin(xy)x))$$
$$= 2\mathrm{e}^{2y} - \cos(xy) + xy\sin(xy)$$
$$\frac{\partial N}{\partial y} = 2\mathrm{e}^{2y} - (\cos(xy) + x(-\sin(xy))y)$$
$$= 2\mathrm{e}^{2y} - \cos(xy) + xy\sin(xy)$$
$$f(x,y) = \int M(x,y) + h(y)$$
$$= \int (\mathrm{e}^{2y} - y\cos(xy))\ \mathrm{d}x + h(y)$$
$$= x\mathrm{e}^{2y} - (y + \sin(xy)\frac{1}{y}) + h(y)$$
$$= x\mathrm{e}^{2y} - \sin(xy) + h(y)$$
$$\frac{\partial f}{\partial y} = 2x\mathrm{e}^{2y} - \cos(xy)x + h'(y) = N(x,y)$$
$$h'(y) = 2y$$
$$h(y) = y^2 + c$$
$$\therefore f(x,y) = x\mathrm{e}^{2y} - \sin(xy) + y^2 + c$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech