

# Multivariable and Vector Calculus

Alvin Lin

August 2017 - December 2017

## Review 2

Arc Length:

$$\int_C f \, dS = \int_a^b f|r'(t)| \, dt$$
$$\int_{C_{AB}} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} \, dS = \int_a^b \vec{F} \cdot \overrightarrow{r'(t)} \, dt$$

For  $r(u, v)$   $(u, v) \in D$ :

$$\iint_S f \, dS = \iint_D f \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| \, du \, dv$$
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \vec{r}) \, dS = \iint_D \vec{F} \cdot \left( \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right) \, du \, dv$$
$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \iint_S \text{curl } F \, dS = \iint_D (\vec{\nabla} \times \vec{F}) \cdot \left( \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right) \, du \, dv$$

Greene's Theorem:

$$\iint_D Q_x - P_y \, dA$$

### Example

Compute the line integral:

$$\int_C x \, dS$$

$C$  is the curve  $y = x^2$  from  $(0,0)$  to  $(1,1)$ .

$$\begin{aligned}\overrightarrow{r(t)} &= \langle t, t^2 \rangle \\ \overrightarrow{r'(t)} &= \langle 1, 2t \rangle \\ \int_C x \, dS &= \int_a^b f |r'(t)| \, dt \\ &= \int_0^1 t \sqrt{1 + 4t^2} \, dt\end{aligned}$$

### Example

Compute the line integral:

$$\int_C \langle y, x + y^2 \rangle \cdot dr$$

$C$  is the curve  $4x^2 + 9y^2 = 36$  in the counterclockwise orientation.

$$\begin{aligned}\overrightarrow{r(t)} &= \left\langle \frac{6 \cos(t)}{2}, \frac{6 \sin(t)}{3} \right\rangle \\ \int_C \langle y, x + y^2 \rangle \cdot dr &= \int_0^{2\pi} \langle 2 \sin(t), 3 \cos(t) + 4 \sin^2(t) \rangle \cdot \langle -3 \sin(t), 2 \cos(t) \rangle \, dt \\ &= \int_0^{2\pi} -6 \sin^2(t) + 6 \cos^2(t) + 8 \sin^2(t) \cos(t) \, dt\end{aligned}$$

Using Greene's Theorem:

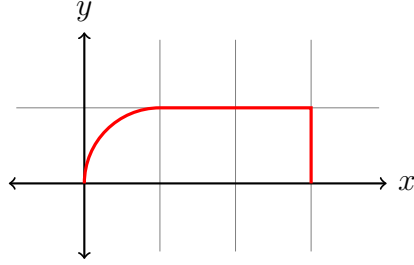
$$\begin{aligned}\int_C \langle y, x + y^2 \rangle \cdot dr &= \iint_D Q_x - P_y \, dA \\ &= \iint_D 1 - 1 \, dA \\ &= 0\end{aligned}$$

### Example

Find the work done along the curve  $C$  of the function:

$$\int_C \langle 2xy, x^2 \rangle \cdot dr$$

given that the curve  $C$  is:



**Example**

Evaluate:

$$\int_C \langle e^y, xe^y + e^z, ye^z \rangle \cdot dr$$

Given  $C$  is the line from  $(0,2,0)$  to  $(4,0,3)$ .

$$\begin{aligned} f_x = e^y & \quad f = xe^y + C_1(y, z) \\ f_y = xe^y + e^z & \quad f = xe^y + ye^z + C_2(x, z) \\ f_z = ye^z & \quad f = ye^z + C_3(x, y) \\ f & = xe^y + ye^z \end{aligned}$$

$$\int_C \langle e^y, xe^y + e^z, ye^z \rangle \cdot dr = \int_{(0,2,0)}^{(4,0,3)} xe^y + ye^z \, dx \, dy \, dz$$

**Example**

Give the parameterization of the region bounded by:

$$x^2 + y^2 + z^2 = 10 \quad z = 1 \quad z = 3$$

$$\vec{r}(R, \theta) = \langle R \cos \theta, R \sin \theta, \sqrt{10 - R^2} \rangle \quad 1 \leq R \leq 3 \quad 0 \leq \theta \leq 2\pi$$

**Example**

Find the surface area of the region bounded by:

$$z = \sqrt{x^2 + y^2} \quad y = x \quad y = x^2$$

$$\begin{aligned}
\overrightarrow{r(x,y)} &= \langle x, y, \sqrt{x^2 + y^2} \rangle \\
\iint_S 1 \, dS &= \int_0^1 \int_{x^2}^x \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| \, dx \, dy \, dx \\
&= \int_0^1 \int_{x^2}^x \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{2x}{2\sqrt{x^2+y^2}} \\ 0 & 1 & \frac{2y}{2\sqrt{x^2+y^2}} \end{vmatrix} \, dy \, dx \\
&= \int_0^1 \int_{x^2}^x \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} \, dy \, dx \\
&= \int_0^1 \int_{x^2}^x \sqrt{2} \, dy \, dx
\end{aligned}$$

### Example

Evaluate:

$$\iint_S \langle x^2, xy, z \rangle \cdot dS$$

where  $S$  is bounded by  $z = x^2 + y^2$  below  $z = 1$  with upwards orientation.

$$\begin{aligned}
\overrightarrow{r(R,\theta)} &= \langle R \cos \theta, R \sin \theta, R^2 \rangle \\
\left| \frac{\partial r}{\partial R} \times \frac{\partial r}{\partial \theta} \right| &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \theta & \sin \theta & 2R \\ -R \sin \theta & R \cos \theta & 0 \end{vmatrix} \\
&= \langle -2R^2 \cos \theta - 2R^2 \sin \theta, R \rangle \\
\iint_S \langle x^2, xy, z \rangle \cdot dS &= \int_0^{2\pi} \int_0^1 \langle R^2 \cos^2 \theta, R^2 \sin \theta \cos \theta, R^2 \rangle \cdot \left| \frac{\partial r}{\partial R} \times \frac{\partial r}{\partial \theta} \right| \, dR \, d\theta \\
&= \int_0^{2\pi} \int_0^1 -2R^4 \cos^3 \theta - 4R^4 \sin^2 \theta \cos \theta + R^3 \, dR \, d\theta
\end{aligned}$$

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)